

Guide to the Atlas Software: Computational Representation Theory of Real Reductive Groups

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In memory of Fokko du Cloux

1 Introduction

The Atlas of Lie Groups and Representations is a project in the representation theory of real reductive groups. The main goal of the `atlas` computer software, currently under development, is to compute the unitary dual of any real reductive Lie group G . As a step in this direction it currently computes the admissible representations of G .

The underlying mathematics of the software is described in *Algorithms for Representation Theory of Real Reductive Groups* [1]. See Sections 1 and 2 of [1] for an overview of the algorithm. This paper is a complement, and consists of a guide to the software illustrated by numerous examples.

The software is currently in an early stage of development (version 0.3 as of April 2008). It is available from the Atlas web site www.liegroups.org.

The `help` command in the `atlas` software is another useful source of information. Also see the `software` section of www.liegroups.org, including `examples` and `Tables of Structure and Representation Theory`. We plan to publish a manual for the software at the time version 1.0 is released.

Marc van Leeuwen is currently working on the *interpreter*, which provides simpler and more powerful input and output methods for the software. This is included in the current software distribution; see the `makefile` and the `source/interpreter` directory. We do not discuss this here since it is in a

state of flux. The input methods described in this paper will be available indefinitely. The interpreter will provide additional output routines, but there should only be minor changes to the output described here.

The `atlas` software was written by Fokko du Cloux. Alfred Noel has also contributed to it. In addition to writing the interpreter, Marc van Leeuwen has made substantial changes to the software, and since version 0.3 has been in charge of software development.

This paper grew out of lectures at the conference in honor of Dragan Milicic and Bill Casselman at Snowbird, July 2006. Bill had a catalytic effect on the Atlas project, which got started at a conference he organized in Montreal in 2002.

1.1 How to read this paper

You should start by downloading the `atlas` software from the Atlas web site www.liegroups.org and installing it. Currently it runs under unix, including Solaris and linux, Mac OSX, and Windows. You should also have the paper *Algorithms for Representation Theory of Real Reductive Groups* handy.

A good approach to learning the software and the *Algorithms* paper is go through this paper, and do the examples using the software. Each section begins with a brief summary of the relevant material from *Algorithms*. We begin very simply, and build up to more complicated examples by the end. We assume some familiarity with the theory of real reductive groups, for example see Chapters 1-8 of [2].

2 Defining basic data

The starting point of the algorithm is a complex reductive algebraic group G together with an *inner class of real forms* of G . The latter is determined by an involution $\gamma \in \text{Out}(G)$, the group of outer automorphisms of G . We refer to (G, γ) as *basic data* [1, Section 4].

We define a semidirect product $G^\Gamma = G \rtimes \Gamma$, where $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$. We let σ act on G by a “distinguished” involution $\tau \in \text{Aut}(G)$ mapping to $\gamma \in \text{Out}(G)$, via the map $\text{Aut}(G) \rightarrow \text{Out}(G)$. (A distinguished involution is one which fixes a splitting datum $(B, H, \{X_\alpha\})$; it is the “most

compact" involution in this inner class.) Write $G^\Gamma = \langle G, \delta \rangle$ where $\delta = 1 \times \sigma$. See [1, Section 5].

A *real form* of G in this inner class is a conjugacy class of involutions $\theta \in \text{Aut}(G)$ mapping to $\gamma \in \text{Out}(G)$. If θ is an involution of G let σ be an antiholomorphic involution of G commuting with θ . Then $G(\mathbb{R}) = G^\sigma$ is a real group in the usual sense, and $G(\mathbb{R})^\theta$ is a maximal compact subgroup of $G(\mathbb{R})$. See [1, Section 5].

The reader should keep in mind the basic example of $\gamma = 1$, so $G^\Gamma = G \times \Gamma$. This is known as the *compact* or *equal rank* inner class; it contains the compact real form of G , and all groups in this inner class contain a compact Cartan subgroup.

The first step in using the `atlas` software is to define basic data (G, γ) . This proceeds in three steps:

- (1) Define a complex reductive Lie algebra \mathfrak{g} , and let G^* be the product of a complex torus and a simply connected, semisimple complex group with Lie algebra \mathfrak{g} ;
- (2) Choose a finite subgroup A of $Z(G^*)$, and set $G = G^*/A$;
- (3) Choose an inner class of real forms of G .

All three steps are accomplished by the `type` command. We break this up into three steps.

2.1 Defining \mathfrak{g} and G^*

A complex reductive Lie algebra is given by a list of types `A_n, B_n, ..., E_8, T_n`, where `T_n` is the abelian Lie algebra \mathbb{C}^n . In response to the `type` command, the software asks for the `Lie type:`. The user then enters such a list, with terms separated by a period. The order is irrelevant here, although it plays a role in steps (2) and (3). The entry `T_2` is the same as `T_1.T_1`.

This defines the complex reductive Lie algebra \mathfrak{g} , and group G^* . Here are some simple examples.

Example 2.1 We start with $SL(2, \mathbb{C})$:

```
main: type
Lie type: A1
```

Here is $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$:

```
main: type
Lie type: A1.A1
```

Here is $SL(2, \mathbb{C}) \times \mathbb{C}^\times$:

```
main: type
Lie type: A1.T1
```

and finally $SL(2, \mathbb{C})^2 \times Spin(5, \mathbb{C}) \times Sp(4, \mathbb{C}) \times Sp(6, \mathbb{C}) \times \mathbb{C}^{\times 4}$:

```
main: type
Lie type: A1.T1.B2.C2.T1.C3.T2.A1
```

2.2 Defining a complex group G

The second step is to pick a finite subgroup of the center Z^* of G^* .

The center of each simple factor of G is a finite cyclic group except in type D_{2n} in which case it is $\mathbb{Z}_2 \times \mathbb{Z}_2$. The finite cyclic group of order n is denoted \mathbb{Z}/n , and is viewed as the group

$$\frac{1}{n}\mathbb{Z}/\mathbb{Z} = \left\{ \frac{0}{n}, \frac{1}{n}, \dots, \frac{n-1}{n} \right\}.$$

The elements of finite order in \mathbb{C}^\times are isomorphic to \mathbb{Q}/\mathbb{Z} , and an element of this group is given by an element of \mathbb{Q} .

Once the user has given the Lie type, the software prompts the user for a finite subgroup of Z^* , generated by a set of elements of Z^* . Each such element is given by a list of fractions, one for each term in the Lie type (two for D_{2n}), separated by commas. Each element is given on a single line; the empty line terminates this aspect of the input. For example simply typing `return` in response to this prompt takes $A = 1$ and $G = G^*$. Typing `sc` has the same effect. Typing `ad` gives the adjoint group (actually a torus times the adjoint group of the derived group of G^*).

Example 2.2 Here is the group $SL(2, \mathbb{C})$:

```
main: type
Lie type: A1
elements of finite order in the center of the simply connected group:
```

```

Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
sc
enter inner class(es):

```

Entering a carriage return alone also gives $SL(2, \mathbb{C})$.

Example 2.3 To define $PSL(2, \mathbb{C})$ take A to be the center of $SL(2, \mathbb{C})$, which is generated by the element of order 2:

```

Lie type: A1
elements of finite order in the center of the simply connected group:
Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2

enter inner class(es):

```

Entering ad instead of 1/2 has the same effect.

Example 2.4 Here is $SO(10, \mathbb{C})$, which is the quotient of $Spin(10, \mathbb{C})$ by the element of Z^* of order 2:

```

main: type
Lie type: D5
elements of finite order in the center of the simply connected group:
Z/4
enter kernel generators, one per line
(ad for adjoint, ? to abort):
2/4

```

Often a reductive group is a quotient of G^* by a diagonal subgroup.

Example 2.5 Here is $GL(2, \mathbb{C}) \simeq SL(2, \mathbb{C}) \times \mathbb{C}^* / \{(-I, -1)\}$:

```

main: type
Lie type: A1.T1
elements of finite order in the center of the simply connected group:
Z/2.Q/Z
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,1/2

```

The center of $Spin(4n, \mathbb{C})$ is not cyclic, and therefore requires two terms.

Example 2.6 Here is $SO(8, \mathbb{C}) \simeq Spin(8, \mathbb{C})/A$ where A is the diagonal subgroup of $Z^* = \mathbb{Z}_2 \times \mathbb{Z}_2$:

```

main: type
Lie type: D4
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,1/2

```

Example 2.7 We can also take the quotient of $Spin(8, \mathbb{C})$ by a non-diagonal \mathbb{Z}_2 subgroup:

```

main: type
Lie type: D8
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,0/2

```

This is not isomorphic to $SO(8, \mathbb{C})$.

This defines the group $G = G^*/A$ and completes step 2.

2.3 Defining an inner class of real forms

We next define an inner class of real forms. Recall this is determined by an involution in $\text{Out}(G)$.

The trivial element of $\text{Out}(G)$ corresponds to the inner class of real forms containing a compact Cartan subgroup. This is the *compact* inner class, and is denoted \mathfrak{c} , and also \mathfrak{e} for *equal rank*. In particular if G has a torus factor its real points are $S^1 \times \cdots \times S^1$.

Another natural inner class is that of the split real form. This class is denoted \mathfrak{s} ; for a torus the real points are $\mathbb{R}^\times \times \cdots \times \mathbb{R}^\times$. In many cases the classes \mathfrak{c} and \mathfrak{s} are the same, for example if $\text{Out}(G) = 1$.

Now suppose $G = G_1 \times G_1$. The outer automorphism switching the two factors corresponds to the inner class of the real form $G_1(\mathbb{C})$, viewed as a real group. This class is denoted \mathfrak{C} ; in the case of a torus this gives $\mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$ (viewed as a real group).

If G is simple, then except in type D_{2n} the classes $\mathfrak{c}, \mathfrak{s}$ exhaust every inner class. In type D_{2n} the classes \mathfrak{c} and \mathfrak{s} coincide, and there is another inner class denoted \mathfrak{u} , for *unequal rank*. See example 2.14. This class is allowed in any type for which the Dynkin diagram has a non-trivial automorphism.

Now suppose $G = G^*$. An inner class of real forms of G is specified by choosing $\mathfrak{c}, \mathfrak{e}, \mathfrak{s}, \mathfrak{u}$ for each simple or T^1 factor, or \mathfrak{C} for each pair of identical (simple or T^1) factors, or factor of type T^{2n} . In general an inner class of real forms of G is given by an *allowed* inner class of real forms of G^* : the involutions in this inner class must factor to G .

To summarize, to specify an inner class of real forms of G , give a list of choices for each simple or torus factor, or pair of identical entries in the case of C :

- \mathfrak{c} : compact
- \mathfrak{e} : equal rank (same as \mathfrak{c})
- \mathfrak{s} : split
- \mathfrak{u} : unequal rank
- \mathfrak{C} : complex (for an identical pair of entries).

The order of the choices corresponds to the order in which the simple and torus factors of G^* were specified. If $G \neq G^*$ some choices may not be allowed.

Here are some examples. Getting slightly ahead of ourself, the `showrealforms` command lists the real forms in the given inner class. In the terminology of `atlas` these are *weak* real forms; see Sections 3 and 4. Also note that the first three inputs to the `type` command can be entered on a single line, provided the second input is `ad` or `sc`.

Example 2.8 The group $SL(2, \mathbb{C})$ has two real forms $SL(2, \mathbb{R})$ and $SU(2)$, both of which are in the same inner class. Thus `c=e=s` in this case.

```
empty: showrealforms
Lie type: A1 sc s
(weak) real forms are:
0: su(2)
1: sl(2,R)
```

This example also illustrates a general principle of the software: entering a command such as `showrealforms` which requires other input is allowed; the system will prompt the user for the missing information.

Example 2.9 The smallest simple group with two inner classes is type A_2 . Here is the split inner class, with only one real form:

```
empty: showrealforms
Lie type: A2 sc s
(weak) real forms are:
0: sl(3,R)
```

Here is the compact inner class, with two real forms:

```
empty: showrealforms
Lie type: A2 sc c
(weak) real forms are:
0: su(3)
1: su(2,1)
```

Example 2.10 Here is $SL(2, \mathbb{C})$ viewed as a real group:

```

empty: showrealforms
Lie type: A2.A2 sc C
(weak) real forms are:
0: sl(3,C)

```

Example 2.11 If the group is a product specify the inner class as a list. For example here is the inner class of $SU(3) \times SL(3, \mathbb{R})$:

```

empty: showrealforms
Lie type: A2.A2 sc cs
(weak) real forms are:
0: su(3).sl(3,R)
1: su(2,1).sl(3,R)

```

(The `sc` means simply connected, and `cs` means compact×split).

Example 2.12 Here is the inner class of $GL(2, \mathbb{R})$:

```

empty: showrealforms
Lie type: A1.T1
elements of finite order in the center of the simply connected group:
Z/2.Q/Z
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,1/2

enter inner class(es): ss
(weak) real forms are:
0: su(2).gl(1,R)
1: sl(2,R).gl(1,R)

```

Group 1 is $GL(2, \mathbb{R})$ (group 0 is the multiplicative group of the quaternions).

Example 2.13 On the other hand here is the inner class of $U(1, 1)$:

```

main: showrealforms
Lie type: A1.T1
elements of finite order in the center of the simply connected group:
Z/2.Q/Z
enter kernel generators, one per line

```

```
(ad for adjoint, ? to abort):  
1/2,1/2
```

```
enter inner class(es): cc  
(weak) real forms are:  
0: su(2).u(1)  
1: sl(2,R).u(1)
```

Groups 0 and 1 are $U(2)$ and $U(1,1)$, respectively.

Example 2.14 The group $SO(12, \mathbb{C})$ has the following real forms, in two inner classes. One inner class consists of the groups $SO(12, 0)$, $S(10, 2)$, $SO(8, 4)$, $SO(6, 6)$ and $SO^*(12)$. This is both the compact inner class (it contains $SO(12, 0)$) and split (it contains $SO(6, 6)$), and is therefore the class $c=e=s$. There is another inner class consisting of $SO(11, 1)$, $SO(9, 3)$ and $SO(7, 5)$. This is the inner class u of unequal rank. This is an example where u is needed.

Here is the compact and split inner class:

```
empty: showrealforms  
Lie type: D6  
elements of finite order in the center of the simply connected group:  
Z/2.Z/2  
enter kernel generators, one per line  
(ad for adjoint, ? to abort):  
1/2,1/2
```

```
enter inner class(es): e  
(weak) real forms are:  
0: so(12)  
1: so(10,2)  
2: so*(12)[1,0]  
3: so*(12)[0,1]  
4: so(8,4)  
5: so(6,6)
```

For an explanation of the two version of $so^*(12)$ see Example 3.3.

The same input with u in place of e gives the unequal rank inner class:

```

enter inner class(es): u
(weak) real forms are:
0: so(11,1)
1: so(9,3)
2: so(7,5)

```

Example 2.15 Here is an example in which the inner class for G^* is not defined for G :

```

main: type
Lie type: A1.A1
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,0/2

```

```

enter inner class(es): C
sorry, that inner class is not compatible with the weight lattice

```

What this means is: the automorphism which switches the two factors in $G^* = SL(2, C) \times SL(2, C)$ does not preserve A , and so does not factor to $G = PSL(2, C) \times SL(2, C)$.

Example 2.16 A more subtle example is the non-diagonal quotient of $Spin(8, C)$ of Example 2.7:

```

main: type
Lie type: D4
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,0/2

```

```

enter inner class(es): u
sorry, that inner class is not compatible with the weight lattice

```

3 Defining a real group

We suppose the user has completed the `type` command, which defines G and an inner class of real forms. We next specify a particular real form of G in the inner class. We discuss strong real forms in the next section.

The command `showrealforms`, discussed in the preceding section, gives a list of the real forms of \mathfrak{g} in the given inner class; the command `realform` gives the same list, and the user can choose the real form from the list.

The real forms of the classical Lie algebras are given in the usual notation. Some examples are `sl(4,R)`, `su(3,1)`, `su(4)`, `sl(4,H)`, `sp(3,2)`, `so(3,2)`, and `so*(10)`. For a torus the real form is specified by `gl(1,R)`, `u(1)` or `gl(1,C)`.

For each exceptional group the real form is specified by specifying the type of the maximal compact subgroup, except that the split real form is denoted `R`. For example the real forms of E_7 are `e7`, `e7(e6.u(1))`, `e7(so(12).su(2))` and `e7(R)`; otherwise known as *compact*, *Hermitian*, *quaternionic* and *split*, respectively.

Example 3.1 Here is the group $SL(2, \mathbb{R})$:

```
empty: realform
Lie type: A1 sc s
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
```

Example 3.2 Here is $PSL(2, \mathbb{R}) \simeq PGL(2, \mathbb{R}) \simeq SO(2, 1)$.

```
main: type
Lie type: A1 ad s
main: realform
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
```

Note that the real forms listed don't depend on the fact that the complex group is $PSL(2, \mathbb{C})$ here instead of $SL(2, \mathbb{C})$; the real form is determined by a real form of \mathfrak{g} .

Example 3.3 The equal rank forms of $SO(12, \mathbb{C})$ were given in Example 2.14:

```
0: so(12)
1: so(10,2)
2: so*(12)[1,0]
3: so*(12)[0,1]
4: so(8,4)
5: so(6,6)
```

The two copies of $\mathfrak{so}^*(12)$ illustrate a technical point. We say two real forms are equivalent if they are conjugate by G . See [1, Section 5]. In the literature equivalence is defined to be conjugacy by $\text{Aut}(G)$. If G is simple the two notions agree in most cases.

There are two real forms (in our sense) of $SO(12, \mathbb{C})$ corresponding to the real group $SO^*(12)$. In other words there are two isomorphic subgroups of $SO(12, \mathbb{C})$, which are not conjugate to each other. They *are* related by an outer automorphism of $SO(12, \mathbb{C})$. (In the Kac classification of real forms each is labelled by a 2 on one of the branches of the fork in the Dynkin diagram). These two groups are denoted $SO^*(12)[1, 0]$ and $SO^*(12)[0, 1]$. The fact that there are two copies of $SO^*(12)$ manifests itself in the representation theory of $SO(p, q)$. See Example 8.8.

In fact we can even see this distinction in terms of the structure theory of the groups themselves. If $G = Spin(12, \mathbb{C})$, $SO(12, \mathbb{C})$ or $PSO(12, \mathbb{C})$ the two real groups locally isomorphic to $SO^*(12)$ are isomorphic, and interchanged by an outer automorphism of G . However in the non-diagonal quotient of $Spin(12, \mathbb{C})$ this fails, as is illustrated in the next example.

Example 3.4 Here are two versions of $SO^*(12)$ in the non-diagonal quotient of $Spin(12, \mathbb{C})$:

```
main: type
Lie type: D6
elements of finite order in the center of the simply connected group:
Z/2.Z/2
enter kernel generators, one per line
(ad for adjoint, ? to abort):
1/2,0/2
```

```

enter inner class(es): c
main: realform
(weak) real forms are:
0: so(12)
1: so(10,2)
2: so*(12)[1,0]
3: so*(12)[0,1]
4: so(8,4)
5: so(6,6)
enter your choice: 2
real: components
component group is (Z/2)^1
real: realform
(weak) real forms are:
0: so(12)
1: so(10,2)
2: so*(12)[1,0]
3: so*(12)[0,1]
4: so(8,4)
5: so(6,6)
enter your choice: 3
real: components
group is connected

```

These two groups are *not* isomorphic. This illustrates the command `components`, which gives the component group of the real group (an elementary abelian two-group).

In D_4 the situation is even more interesting. Let $G = Spin(8, \mathbb{C})$ and consider the three real forms, $Spin(6, 2)$ and the two versions of $Spin^*(8)$. In fact these are isomorphic; $Spin(6, 2) \simeq Spin^*(8)$, and these three groups are interchanged by the outer automorphism group of $Spin(8, \mathbb{C})$, which is S_3 . Similar statements hold in $PSO(8, \mathbb{C})$. We leave it to the reader to analyze real forms of $SO(8, \mathbb{C})$ and the non-diagonal quotient of $Spin(8, \mathbb{C})$, in which this symmetry is broken in different ways. See Examples 2.6 and 2.7.

In general in $SO(4n, \mathbb{C})$ (type D_{2n}) there are two non-conjugate copies of $SO^*(4n)$. However in $SO(4n+2, \mathbb{C})$ (type D_{2n+1}) all copies of $SO^*(4n+2)$ are in fact conjugate by $SO(2n+2, \mathbb{C})$. This is illustrated by the next example.

Example 3.5 Here are the equal rank real forms of $SO(10, \mathbb{C})$:

```

empty: showrealforms
Lie type: D5
elements of finite order in the center of the simply connected group:
Z/4
enter kernel generators, one per line
(ad for adjoint, ? to abort):
2/4

enter inner class(es): c
(weak) real forms are:
0: so(10)
1: so(8,2)
2: so*(10)
3: so(6,4)

```

Note that there is only one real form $SO^*(10)$.

4 The Spaces \mathcal{X} and \mathcal{X}^r

Fix basic data (G, γ) as in Section 2.

By a *strong involution* for (G, γ) we mean an element $\xi \in G^\Gamma \backslash G$ satisfying $\xi^2 \in Z$ ($Z = Z(G)$ is the center of G). Let \mathcal{I} be the set of strong involutions. A *strong real form* is an equivalence class of strong involutions (equivalence is by conjugacy by G). If ξ is a strong involution then $\theta_\xi = \text{int}(\xi)$ is an involution of G in the inner class of γ , and this gives a surjective map from strong real forms to real forms in this inner class. If G is adjoint this map is bijective, but not otherwise. Let $K_\xi = G^{\theta_\xi}$; this is the complexified maximal compact subgroup of the corresponding real group $G(\mathbb{R})$. See Section 2.

Recall (Section 2) $\delta = 1 \times \sigma$ is the distinguished element of G^Γ . We fix once and for all a Cartan subgroup H of G stable by $\text{int}(\delta)$. Define

$$(4.1) \quad \begin{aligned} \tilde{\mathcal{X}} &= \{g \in \text{Norm}_{G^\Gamma \backslash G}(H) \mid g^2 \in Z\} \\ \mathcal{X} &= \tilde{\mathcal{X}}/H \end{aligned}$$

(the quotient is by the conjugation action of H). See [1, Section 9]. The space \mathcal{X} is our main combinatorial object, known as the *(one sided) parameter space*.

Mathematically the space \mathcal{X} is a natural object. From the point of view of computations it has the disadvantage that (if G is not semisimple) it may be infinite, and the fibers of the map from strong real forms to real forms may be infinite. The software works exclusively with the *reduced parameter space* \mathcal{X}^r which we now describe. See [1, Section 13].

Let $\theta = \text{int}(\delta)$ and choose a set $Z^r \subset Z$ of representatives of

$$(4.2) \quad \{z \in Z \mid \theta(z) = z\} / \{z\theta(z) \mid z \in Z\}.$$

Define reduced versions of \mathcal{I} , $\tilde{\mathcal{X}}$ and \mathcal{X} :

$$(4.3) \quad \begin{aligned} \mathcal{I}^r &= \{g \in G^\Gamma \backslash G \mid g^2 \in Z^r\} \\ \tilde{\mathcal{X}}^r &= \{g \in \text{Norm}_{G^\Gamma \backslash G}(H) \mid g^2 \in Z^r\}, \mathcal{X}^r = \tilde{\mathcal{X}}^r / H. \end{aligned}$$

Henceforth we define a strong real form in this revised sense: \mathcal{I}^r is the space of strong real forms, with equivalence given by conjugacy by G as before. The map from strong real forms to real forms is still surjective. The fibers of this map are finite, and \mathcal{X}^r is a finite set.

Choose a set $\{\xi_i \mid 1 \leq i \leq n\} \subset \tilde{\mathcal{X}}$ of representatives of strong real forms (in the sense of the reduced parameter space). That is

$$(4.4) \quad \{\xi_1, \dots, \xi_n\} \xrightarrow{1-1} \mathcal{I}^r / G.$$

For each i let x_i be the image of ξ_i in \mathcal{X}^r , $\theta_i = \text{int}(\xi_i)$ and $K_i = G^{\theta_i}$. See [1, 5.15].

Let $W^\Gamma = \text{Norm}_{G^\Gamma}(H)/H$ and

$$\mathcal{I}_W = \{\tau \in W^\Gamma \backslash W \mid \tau^2 = 1\}.$$

We continue to write δ for the image of $\delta \in G^\Gamma$ in W^Γ , and θ for the involution $\text{int}(\delta)$ of W . Then $W^\Gamma = \langle W, \delta \rangle$. By a *twisted involution* in W we mean an element $w \in W$ satisfying $w\theta(w) = 1$, and we say two such elements w, w' are *twisted-conjugate* if $w' = yw\theta(y^{-1})$ for some $y \in W$. The map $w \rightarrow w\delta$ is a bijection between the space of twisted involutions and \mathcal{I}_W , taking twisted-conjugacy to ordinary conjugacy by W . We refer to \mathcal{I}_W as the space of twisted involutions, and pass back and forth between the two notions. See [1, 9.14].

There is a natural surjective map $p : \mathcal{X}^r \rightarrow \mathcal{I}_W$ [1, Lemma 9.12]. The fiber over an element $\tau \in \mathcal{I}_W$ is denoted \mathcal{X}_τ^r .

The group $\text{Norm}_G(H)$ acts by conjugation on $\tilde{\mathcal{X}}^r$, and this factors to an action of W on \mathcal{X}^r . Because of its relationship to the cross action of [3, Definition 8.3.1] we call this the *cross action* of W , and denote it $w \times x$. This action is equivariant for p and the conjugation action of W on \mathcal{I}_W : $p(w \times x) = wp(x)w^{-1}$.

5 Cartan subgroups

Fix basic data (G, γ) . Let θ_{qs} be a quasisplit involution in this inner class and let $K_{qs} = G^{\theta_{qs}}$. Thus K_{qs} is the complexified maximal compact subgroup of the quasisplit form $G_{qs}(\mathbb{R})$ of G . The conjugacy classes of Cartan subgroups of $G_{qs}(\mathbb{R})$, equivalently the K_{qs} -conjugacy classes of θ_{qs} -stable Cartan subgroups of G , are in natural bijection with \mathcal{I}_W/W , conjugacy classes of twisted involutions in W . The conjugacy classes of Cartan subgroups for any real form of G are a subset of those for the quasisplit real form. See [1, Propositions 12.9 and 12.12].

The `cartan` command gives a list of Cartan subgroups for a given inner form. For each Cartan subgroup it gives the following information. See the `help` file for the `cartan` command for more details.

First it gives the structure of the Cartan subgroup as a real torus: $H(\mathbb{R})$ is isomorphic to $(\mathbb{R}^\times)^a \times (S^1)^b \times (\mathbb{C}^\times)^c$ for integers (a, b, c) ; these are the `split`, `compact` and `complex` entries in the output of `cartan`.

Each conjugacy class of twisted involutions contains a unique canonical representative $\tau = w\delta$. The next line of output is a reduced expression for w .

The first entry on the next line is the number of twisted involutions in this conjugacy class. For an explanation of `fiber rank` see and `#X_r` see Section 9.

Associated to any Cartan subgroup are the sets of real and imaginary roots (see Section 7), each of which is a root system. These are given by `imaginary root system` and `real root system`, respectively. The `complex factor line` in the output has to do with the Weyl group; see Section 6.

The lines beginning `real form...` give information about \mathcal{X}_τ^r . For now we observe only that for a given Cartan subgroup only the real forms which contain this Cartan subgroup are displayed. See Section 9.1.

Example 5.1 Let $G = SL(2, \mathbb{C})$, which has a unique inner class of real

forms. Then $\mathcal{I}_W = W \times \Gamma$; we can drop Γ and write $\mathcal{I}_W/W = \{1, s\}$. There are two conjugacy classes of Cartan subgroups of $SL(2, \mathbb{R})$, compact and split.

```

empty: cartan
Lie type: A1 sc s
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
Name an output file (return for stdout, ? to abandon):

Cartan #0:
split: 0; compact: 1; complex: 0
canonical twisted involution:
twisted involution orbit size: 1; fiber rank: 1; #X_r: 2
imaginary root system: A1
real root system is empty
complex factor is empty
real form #1: [0] (1)
real form #0: [1] (1)

```

```

Cartan #1:
split: 1; compact: 0; complex: 0
canonical twisted involution: 1
twisted involution orbit size: 1; fiber rank: 0; #X_r: 1
imaginary root system is empty
real root system: A1
complex factor is empty
real form #1: [0] (1)

```

Cartan #0 is always the fundamental (most compact) Cartan subgroup, in this case S^1 . The corresponding twisted involution is the identity, and its orbit is itself. Both real forms contain this Cartan subgroup.

Cartan #1 is the split Cartan subgroup \mathbb{R}^\times , and this occurs only in real form #1, i.e. $SL(2, \mathbb{R})$.

Example 5.2 Here is the complex group $SL(2, \mathbb{C})$:

```

empty: cartan
Lie type: A1.A1 sc C
there is a unique real form: sl(2,C)
Name an output file (return for stdout, ? to abandon):

```

```

Cartan #0:
split: 0; compact: 0; complex: 1
canonical twisted involution:
twisted involution orbit size: 2; fiber rank: 0; #X_r: 2
imaginary root system is empty
real root system is empty
complex factor: A1
real form #0: [0] (1)

```

There is one real form, and one conjugacy class of Cartan subgroups, isomorphic to \mathbb{C}^\times . For a complex group the twisted involutions are in bijection with the Weyl group, so in this case there are 2.

Example 5.3 Here are the Cartan subgroups of $Sp(4, \mathbb{R})$. See [1, Example 14.19].

```

empty: cartan
Lie type: C2 sc s
(weak) real forms are:
0: sp(2)
1: sp(1,1)
2: sp(4,R)
enter your choice: 2
Name an output file (return for stdout, ? to abandon):

```

```

Cartan #0:
split: 0; compact: 2; complex: 0
canonical twisted involution:
twisted involution orbit size: 1; fiber rank: 2; #X_r: 4
imaginary root system: B2
real root system is empty
complex factor is empty
real form #2: [0,1] (2)

```

real form #1: [2] (1)
real form #0: [3] (1)

Cartan #1:
split: 0; compact: 0; complex: 1
canonical twisted involution: 2,1,2
twisted involution orbit size: 2; fiber rank: 0; #X_r: 2
imaginary root system: A1
real root system: A1
complex factor is empty
real form #2: [0] (1)
real form #1: [1] (1)

Cartan #2:
split: 1; compact: 1; complex: 0
canonical twisted involution: 1,2,1
twisted involution orbit size: 2; fiber rank: 1; #X_r: 4
imaginary root system: A1
real root system: A1
complex factor is empty
real form #2: [0] (1)

Cartan #3:
split: 2; compact: 0; complex: 0
canonical twisted involution: 1,2,1,2
twisted involution orbit size: 1; fiber rank: 0; #X_r: 1
imaginary root system is empty
real root system: B2
complex factor is empty
real form #2: [0] (1)

There are four conjugacy classes of Cartan subgroups, isomorphic to $S^1 \times S^1$, \mathbb{C}^\times , $S^1 \times \mathbb{R}^\times$ and $\mathbb{R}^\times \times \mathbb{R}^\times$. All four are contained in the split group $Sp(4, \mathbb{R})$; two of them are contained in $Sp(1, 1)$, and only the compact Cartan subgroup occurs in $Sp(2, 0)$.

Example 5.4 Here is the real form of E_6 with K of type F_4 :

empty: cartan

```

Lie type: E6 sc s
(weak) real forms are:
0: e6(f4)
1: e6(R)
enter your choice: 0
Name an output file (return for stdout, ? to abandon):

Cartan #0:
split: 0; compact: 2; complex: 2
canonical twisted involution:
twisted involution orbit size: 45; fiber rank: 2; #X_r: 180
imaginary root system: D4
real root system is empty
complex factor: A2
real form #1: [0,1,2] (3)
real form #0: [3] (1)

```

This group has a unique conjugacy class of Cartan subgroups. See Example 8.14.

We discuss Cartan subgroups of the Classical groups.

Example 5.5 Type A_{n-1} . It is most convenient to take $G = GL(n, \mathbb{C})$.

First let $\gamma = 1$, so $W^\Gamma = S_n \times \Gamma$. It is well known that the conjugacy classes of involutions in S_n are parametrized by ordered pairs $(a, b) \in \mathbb{N}^2$ satisfying $a + 2b = n$: in cycle notation take $w = (1, 2)(3, 4) \dots (2b - 1, 2b)$.

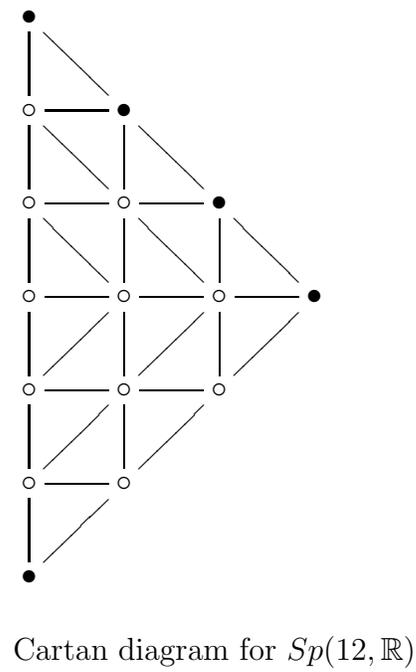
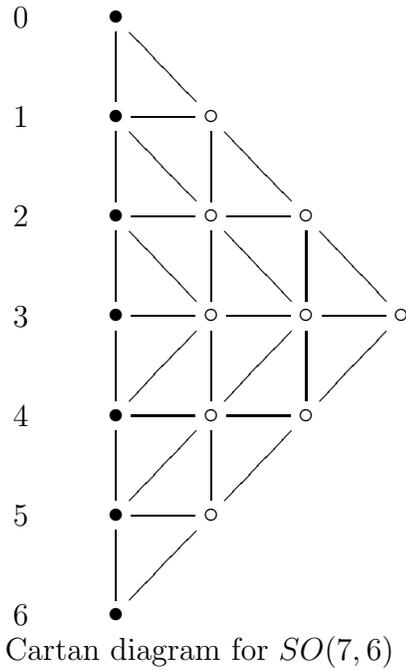
The quasisplit group in this inner class is the unitary group $U(m, m)$ or $U(m + 1, m)$. The Cartan subgroup corresponding to (a, b) is isomorphic to $(S^1)^a \times (\mathbb{C}^\times)^b$; the identity element corresponds to the compact Cartan subgroup.

Now suppose γ is given by the unique non-trivial automorphism of the Dynkin diagram ($n \geq 3$). The split real form is $GL(n, \mathbb{R})$. It turns out that the twisted involutions in W are also parametrized by pairs (a, b) with $a + 2b = n$; in this case the corresponding Cartan subgroup is isomorphic to $(\mathbb{R}^\times)^a \times (\mathbb{C}^\times)^b$. (This is an aspect of Vogan duality; see Section 8 and [1, 1.35 and Corollary 10.9].)

Example 5.6 Types B_n and C_n .

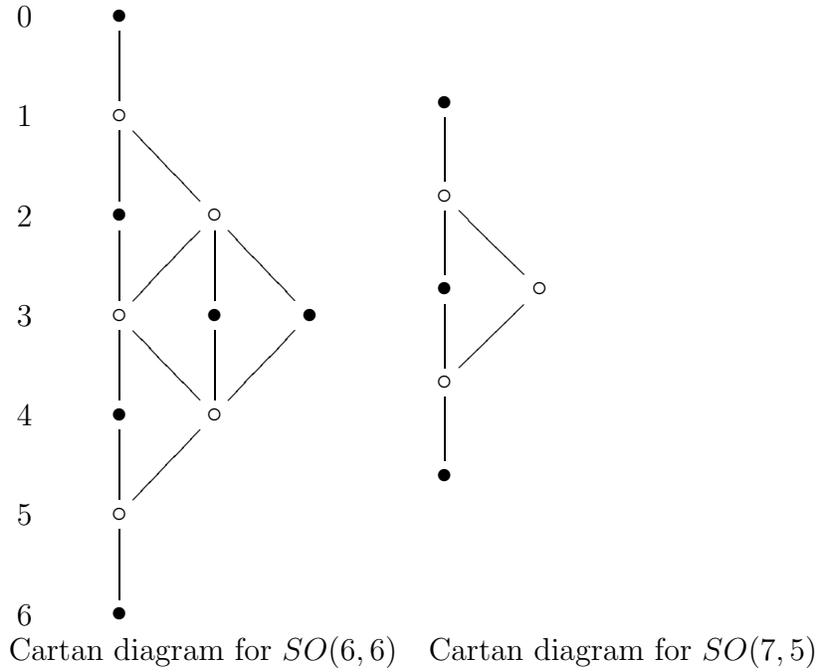
In this case γ is necessarily trivial, and the Cartan subgroups of $SO(n+1, n)$ or $Sp(2n, \mathbb{R})$ are parametrized by conjugacy classes of involutions in $W \simeq S_n \times \mathbb{Z}_2^n$. These are parametrized by (a, b, c) with $a + b + 2c = n$, and the corresponding Cartan subgroup is isomorphic to $(S^1)^a \times (\mathbb{R}^\times)^b \times (\mathbb{C}^\times)^c$.

It is interesting to consider the Hasse diagram of these Cartan subgroups. This is the graph, with one node for each Cartan subgroup, and an edge for each Cayley transform (cf. Section 7) relating two Cartan subgroups. We make a node black if the corresponding Cartan subgroup is the most split Cartan subgroup of a real form of G . The split rank is given in the first column.



Example 5.7 Type D_n .

In this case (for $n \geq 2$) there are two choices of γ , corresponding to the quasisplit groups $SO(n, n)$ and $SO(n + 1, n - 1)$ ($\gamma = 1$ corresponds to $SO(n, n)$ if n is even, and $SO(n + 1, n - 1)$ if n is odd). It is most convenient to group these two real forms together. Then the Cartan subgroups are parametrized by (a, b, c) with $a + b + 2c = n$, except that $(0, 0, c)$ is counted twice. (These two Cartan subgroups are conjugate by the outer automorphism of $SO(n, n)$ coming from $O(n, n)$.) Again the corresponding Cartan subgroup is isomorphic to $(S^1)^a \times (\mathbb{R}^\times)^b \times (\mathbb{C}^\times)^c$. If a is even this Cartan subgroup occurs in $SO(n, n)$ and in $SO(n + 1, n - 1)$ otherwise.



Example 5.8 Note that the Cartan subgroups in $G(\mathbb{R})$ only depend on W^Γ , and are therefore independent of isogeny (this is not true in the p-adic case). While the list of Cartan subgroups is independent of isogeny, the description as a real torus is not, and the nature of the torus can change unexpectedly under isogenies.

For example suppose G is of type $D_2 \simeq A_1 \times A_1$ and $\gamma \neq 1$. There is a unique conjugacy class of Cartan subgroups in this case. In the simply connected case $G(\mathbb{R}) \simeq SL(2, \mathbb{C})$ and the Cartan subgroup is isomorphic to \mathbb{C}^\times (cf. Example 5.2).

For $SO(3,1)$ we get $S^1 \times \mathbb{R}^\times$:

split: 1; compact: 1; complex: 0

and for $PSO(3,1) \simeq PSL(2, \mathbb{C})$ we get \mathbb{C}^\times again:

split: 0; compact: 0; complex: 1

Note that $G(\mathbb{R})$ is a connected complex group if $G(\mathbb{C}) = Spin(4, \mathbb{C})$ or $PSO(4, \mathbb{C})$, but not $SO(4, \mathbb{C})$. In fact we have:

G	$G(\mathbb{R})$	$H(\mathbb{R})$
$Spin(4, \mathbb{C}) \simeq SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$	$Spin(3, 1) \simeq SL(2, \mathbb{C})$	\mathbb{C}^\times
$SO(4, \mathbb{C})$	$SO(3, 1)$	$\mathbb{R}^\times \times S^1$
$PSO(4, \mathbb{C}) \simeq PSL(2, \mathbb{C}) \times PSL(2, \mathbb{C})$	$PSO(3, 1) \simeq PSL(2, \mathbb{C})$	\mathbb{C}^\times

Remark 5.9 If $G(\mathbb{R})$ is a real form of $GL(n, \mathbb{C})$, $SL(n, \mathbb{C})$, $Sp(2n, \mathbb{C})$ or $SO(n, \mathbb{C})$ then two Cartan subgroups are isomorphic if and only if they are conjugate by $G(\mathbb{R})$, or by an automorphism of $G(\mathbb{R})$ in the case of $(\mathbb{C}^\times)^n \subset SO(2n, 2n)$ (cf. Example 5.7). It is perhaps surprising that this fails badly for isogenous groups, as the following example shows.

Example 5.10 Let $G(\mathbb{R}) = Spin(n, n)$ or $PSO(n, n)$ with n even ($n \geq 4$). There are three non-conjugate Cartan subgroups isomorphic to $\mathbb{R}^\times \times S^1 \times (\mathbb{C}^\times)^{\frac{n}{2}-1}$. Two of these are interchanged by an outer automorphism of $G(\mathbb{R})$. (The corresponding Cartan subgroups of $SO(n, n)$ are isomorphic to $(\mathbb{C}^\times)^{\frac{n}{2}}$.) The third one is not related to the others: it has a different real Weyl group (see the `complex factor` entry).

Here are the three Cartan subgroups in question for $Spin(6, 6)$.

```

empty: cartan
Lie type: D6 sc s
(weak) real forms are:
0: so(12)
1: so(10,2)
2: so*(12)[1,0]
3: so*(12)[0,1]
4: so(8,4)
5: so(6,6)
enter your choice: 5
Name an output file (return for stdout, ? to abandon):
...
Cartan #4:
split: 1; compact: 1; complex: 2
canonical twisted involution: 3,4,5,6,4,3,2,3,4,5,6,4,3,1,2,3,4,5,6,4,3,2,1
twisted involution orbit size: 180; fiber rank: 1; #X_r: 360
imaginary root system: A1.A1.A1
real root system: A1.A1.A1
complex factor: A1

```

real form #5: [0] (1)
 real form #4: [1] (1)

Cartan #5:

split: 1; compact: 1; complex: 2
 canonical twisted involution: 6,4,5,3,4,6,2,3,4,5,1,2,3,4,6
 twisted involution orbit size: 60; fiber rank: 1; #X_r: 120
 imaginary root system: A1.A1.A1
 real root system: A1.A1.A1
 complex factor: A2
 real form #5: [0] (1)
 real form #3: [1] (1)

Cartan #6:

split: 1; compact: 1; complex: 2
 canonical twisted involution: 5,4,6,3,4,5,2,3,4,6,1,2,3,4,5
 twisted involution orbit size: 60; fiber rank: 1; #X_r: 120
 imaginary root system: A1.A1.A1
 real root system: A1.A1.A1
 complex factor: A2
 real form #5: [0] (1)
 real form #2: [1] (1)
 ...

A similar phenomenon holds in the split real forms of F_4 and G_2 .

6 Weyl Groups

Fix basic data (G, γ) , $\xi \in \tilde{\mathcal{X}}$, and set $K = G^{\theta_\xi}$ (cf. Section 4). The “real” Weyl group $W(K, H) = \text{Norm}_K(H)/H \cap K$ plays an important role. It is isomorphic to $W(G(\mathbb{R}), H(\mathbb{R})) = \text{Norm}_{G(\mathbb{R})}(H(\mathbb{R}))/H(\mathbb{R})$ where $G(\mathbb{R})$ is a real form of G corresponding to K , and $H(\mathbb{R})$ is the corresponding real form of H . See [1, Section 12].

We briefly recall some constructions from [1, Section 8], also see [4, Proposition 4.16]. Let τ be the image of ξ in \mathcal{I}_W . We have

$$(6.1) \quad W(K, H) \simeq (W_C)^\tau \times (W(M \cap K, H) \times W_\tau).$$

Here

- W_r is the Weyl group of the system of real roots;
- $(W_C)^\tau$ is the Weyl group of a certain root system constructed using complex roots ([4, Proposition 3.12]);
- $W(K \cap M, H) \simeq W_{i,c} \times \mathcal{A}(H) \subset W_i$, where $W_{i,c}$ is the Weyl group of the root system of compact imaginary roots, and $\mathcal{A}(H)$ is a certain elementary abelian two-group.

See Section 7 for the definition of real and imaginary roots.

To describe $W(K, H)$ it is therefore sufficient to describe $W_r, W_{i,c}, (W_C)^\tau$ and $\mathcal{A}(H)$. The first three are Weyl groups of root systems. The `realweyl` command describes these three root systems, denoted `W_r`, `W_ic`, and `W^C` respectively; and $\mathcal{A}(H)$, denoted `A`.

Example 6.2 As usual we start with $SL(2, \mathbb{R})$:

```
empty: realweyl
Lie type: A1 sc s
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
cartan class (one of 0,1): 0
Name an output file (return for stdout, ? to abandon):

real weyl group is W^C.((A.W_ic) x W^R), where:
W^C is trivial
A is trivial
W_ic is trivial
W^R is trivial
```

This is the compact Cartan subgroup, for which $W(K, H)$ is trivial. Here is the split Cartan subgroup:

```
real: realweyl
cartan class (one of 0,1): 1
Name an output file (return for stdout, ? to abandon):

real weyl group is W^C.((A.W_ic) x W^R), where:
```

W^C is trivial
 A is trivial
 W_{ic} is trivial
 W^R is a Weyl group of type A_1

generators for W^R :
1

For the split Cartan subgroup the only non-trivial factor is W_r , and $W(K, H) = W_r \simeq \mathbb{Z}_2$.

Example 6.3 There is a small change when we compute the Weyl group for the compact Cartan subgroup of $PSL(2, \mathbb{R})$ instead:

```

empty: realweyl
Lie type: A1 ad s
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
cartan class (one of 0,1): 0
Name an output file (return for stdout, ? to abandon):

```

```

real weyl group is  $W^C \cdot ((A \cdot W_{ic}) \times W^R)$ , where:
 $W^C$  is trivial
 $A$  is an elementary abelian 2-group of rank 1
 $W_{ic}$  is trivial
 $W^R$  is trivial

```

```

generators for  $A$ :
1

```

In this case $\mathcal{A}(H) = \mathbb{Z}_2$, so $W(K, H) = \mathbb{Z}_2$. Recall $PSL(2, \mathbb{R}) \simeq SO(2, 1)$ is disconnected; the non-trivial Weyl group element is given by an element in the non-identity component.

Example 6.4 Here is an example, for split groups of type D_{2m} , in which the group $\mathcal{A}(H)$ is quite large. We take the Cartan subgroup $(S^1)^2 \times (\mathbb{C}^\times)^{m-1}$

(cf. Example 5.7). In the notation at the beginning of [1, Section 12] we have

$$\begin{aligned}
 \Delta_i &= A_1^{m-1} \times D_2 \simeq A_1^{m+1} \\
 \Delta_r &= A_1^{m-1} \\
 \Delta_C &= A_{m-2} \times A_{m-2} \\
 W_i &\simeq \mathbb{Z}_2^{m+1}, W_r \simeq \mathbb{Z}_2^{m-1} \\
 W_{i,c} &= 1 \\
 (W_C)^\tau &\simeq S_{m-1} \\
 W^\tau &\simeq S_{m-1} \ltimes [\mathbb{Z}_2^{m+1} \times \mathbb{Z}_2^{m-1}].
 \end{aligned}
 \tag{6.5}$$

Here S_{m-1} acts trivially on the final two factors of \mathbb{Z}_2^{m+1} , and W^τ is the centralizer of τ in W ; this group contains $W(K, H)$. So far this discussion is independent of isogeny.

Since $W_{i,c} = 1$ the real Weyl group is the same as W^τ , with the factor $W_i = \mathbb{Z}_2^{m+1}$ replaced by $\mathcal{A}(H) \subset W_i$. The group $\mathcal{A}(H)$ depends on the isogeny. If G is adjoint it is equal to W_i . The smallest possible value of $\mathcal{A}(H)$ occurs when G is simply connected. It isn't as easy to compute $\mathcal{A}(H)$ in this case; the software will tell us.

For example consider the split group $Spin(8, 8)$ of type D_8 ($m = 4$). The Cartan class is #5:

```

Cartan #5:
split: 0; compact: 2; complex: 3
canonical twisted involution: 6,7,8,6,5,[truncated]
twisted involution orbit size: 3360; fiber rank: 2; #X_r: 13440
imaginary root system: A1.A1.A1.A1.A1
real root system: A1.A1.A1
complex factor: A2
...

```

and here is the relevant output from `realweyl`:

```

real weyl group is W^C.((A.W_ic) x W^R), where:
W^C is isomorphic to a Weyl group of type A2
A is an elementary abelian 2-group of rank 3
W_ic is trivial
W^R is a Weyl group of type A1.A1.A1

```

Thus $\mathcal{A}(H) \simeq \mathbb{Z}_2^{m-1}$.

If H is a fundamental (most compact) Cartan subgroup then $H \cap K$ is a Cartan subgroup of K , and it is not hard to see that $W(G, H) \simeq W(K, H \cap K)$. In particular if K is connected this is the Weyl group of the root system of $H \cap K$ in K . It is interesting to consider this in the case of real forms of E_6 of unequal rank.

Example 6.6 First consider $E_6(F_4)$:

```
empty: realweyl
Lie type: E6 sc s
(weak) real forms are:
0: e6(f4)
1: e6(R)
enter your choice: 0
there is a unique conjugacy class of Cartan subgroups
Name an output file (return for stdout, ? to abandon):
```

```
real weyl group is W^C.((A.W_ic) x W^R), where:
W^C is isomorphic to a Weyl group of type A2
A is trivial
W_ic is a Weyl group of type D4
W^R is trivial
```

Thus $W(K, H) \simeq W(A_2) \times W(D_4) \simeq S_3 \times W(D_4)$. The action of S_3 on $W(D_4)$ is induced by the action of S_3 on the Dynkin diagram of D_4 . By the remarks above we have obtained the classical isomorphism

$$(6.7) \quad W(F_4) \simeq S_3 \times W(D_4).$$

Example 6.8 Something similar happens for the split real form of E_6 , in which K is of type C_4 . In this case `realweyl` gives:

```
real weyl group is W^C.((A.W_ic) x W^R), where:
W^C is isomorphic to a Weyl group of type A2
A is an elementary abelian 2-group of rank 2
W_ic is a Weyl group of type A1.A1.A1.A1
W^R is trivial
```

Thus

$$(6.9) \quad \begin{aligned} W(C_4) &\simeq S_3 \times (\mathbb{Z}_2^2 \times \mathbb{Z}_2^4) \\ &\simeq S_4 \times \mathbb{Z}_2^4. \end{aligned}$$

Example 6.10 Complex groups.

Fix basic data $(G = G_1 \times G_1, \gamma)$ for some group G_1 , with γ the inner class of the complex group (γ switches the two factors). Thus $G(\mathbb{R}) = G_1(\mathbb{C})$ and $K = G_1^\Delta$ (the diagonal embedding). If H_1 is a Cartan subgroup of G_1 then $H = H_1 \times H_1$ is a Cartan subgroup of G , $W(G, H) \simeq W(G_1, H_1) \times W(G_1, H_1)$, and $W(K, H) = W(G_1, H_1)^\Delta$.

In the notation of the output of `realweyl` all terms are trivial except $W^{\mathbb{C}}$, which equals $W(G_1, H_1)$.

```
empty: type
Lie type: A3.A3 sc C
main: realweyl
there is a unique real form: sl(4,C)
there is a unique conjugacy class of Cartan subgroups
Name an output file (hit return for stdout):

real weyl group is W^C.((A.W_ic) x W^R), where:
W^C is isomorphic to a Weyl group of type A3
A is trivial
W_ic is trivial
W^R is trivial

generators for W^C:
2,5
3,6
1,4
```

Simple roots 1, 2, 3 give the first copy of G_1 , and 4, 5, 6 give the second. The elements s_1s_4 , s_2s_5 and s_3s_6 generate $W(G_1, H_1)^\Delta = W(A_3) = S_4$.

7 K orbits on the flag variety and \mathcal{X}^r

Fix basic data (G, γ) (Section 2) and let \mathcal{X}^r be the reduced parameter space (Section 4). Given $x \in \mathcal{X}^r$ let

$$\mathcal{X}^r[x] = \{x' \in \mathcal{X}^r \mid x' \text{ is } G\text{-conjugate to } x\}$$

(see [1, 9.7]). Choose a preimage ξ of x in $\tilde{\mathcal{X}}^r$ and let $\theta = \text{int}(\xi)$, $K = G^\theta$. Then $\mathcal{X}^r[x]$ is isomorphic to $K \backslash G/B$. In fact, one of the important properties of \mathcal{X}^r is that it captures information about the K orbits on G/B for all K in this inner class. Recall (4.4) $\{\xi_i \mid 1 \leq i \leq n\}$ is a set of representatives of the strong real forms in this inner class, and for each $i \in I$ we let $K_i = K_{\xi_i}$. Then [1, Corollary 9.9]:

$$\mathcal{X}^r = \prod_{i=1}^n \mathcal{X}[x_i] \simeq \prod_{i=1}^n K_i \backslash G/B.$$

The structure of $K_i \backslash G/B$ is determined to a large extent by the cross action (cf. Section 4), and *Cayley transforms*, which appear in the space \mathcal{X}^r as follows. Fix $x \in \mathcal{X}^r$ and let $\tau = p(x) \in \mathcal{I}_W$. Then τ acts on the roots, and a root α is classified as *imaginary*, *real* or *complex* with respect to τ if $\tau(\alpha) = \alpha$, $-\alpha$ or neither, respectively. Let ξ be a preimage of x in $\tilde{\mathcal{X}}^r$. If α is imaginary we say it is *compact* or *noncompact* with respect to x if this holds with respect to θ_ξ (independent of the choice of ξ). See [1, Section 14] for details.

Now suppose α is a noncompact imaginary root with respect to x . Then associated to α and x is a new element of \mathcal{X}^r denoted $c^\alpha(x)$. It satisfies $p(c^\alpha(x)) = s_\alpha p(x)$, and α is real with respect to $s_\alpha \tau$. Its inverse is the single or double-valued *real* Cayley transform $c_\alpha(x)$: if α is real with respect to x then $c_\alpha(x)$ is a set with one or two elements.

The (finite) set of orbits of K on G/B is described by the `kgb` command.

Example 7.1 We first consider $SL(2, \mathbb{R})$ and $PGL(2, \mathbb{R})$. See [1, Example 12.20].

```
main: kgb
(weak) real forms are:
0: su(2)
1: sl(2,R)
```

```

enter your choice: 1
kgbsize: 3
Name an output file (return for stdout, ? to abandon):
0:  0  0  [n]  1  2
1:  0  0  [n]  0  2
2:  1  1  [r]  2  *  1

```

Here $G(\mathbb{R}) = SL(2, \mathbb{R})$, $K(\mathbb{R}) = S^1$ and $K = K(\mathbb{C}) = \mathbb{C}^\times$. This acts on $G/B = P^1(\mathbb{C}) = \mathbb{C} \cup \infty$ by $\mathbb{C}^\times \ni z : w \rightarrow z^2 w$. The entry `kgbsize` gives the number of orbits, 3 in this case. These are $0, \infty$ and \mathbb{C}^\times , labelled #0,1,2 respectively (the first entry in each line) in the output of `kgb`. (We describe other parts of the output below).

Here is the result for $PGL(2, \mathbb{R})$; see [1, Example 12.25] for more detail.

```

0:  0  0  [n]  0  1
1:  1  1  [r]  1  *  1

```

In this case G/B is the same, but $K = O(2, \mathbb{C})$; an element from the non-identity component identifies 0 and ∞ , so there are only two orbits.

We now explain the other information given in the output. Fix a row `i` with corresponding $x_i \in \mathcal{X}^r[x] \simeq K_\xi \backslash G/B$, and write \mathcal{O}_i for the corresponding orbit of K_ξ on G/B .

The image of x_i in \mathcal{I}_W corresponds to a Cartan subgroup; the first entry gives the number of this Cartan subgroup in the output of the `cartan` command. The second number is the length of this orbit, which is $\dim(\mathcal{O}_i) - \dim(\mathcal{O}_0)$. In the equal rank case ($\gamma = 1$) $\dim(\mathcal{O}_0) = 0$.

The simple roots are labelled $1, \dots, n$. The term in brackets in row `i` of the output gives the *type* of each simple root: `r,C,n,c` for real, complex, noncompact imaginary, or compact imaginary, respectively.

The next n columns give the cross action of the simple roots. An entry `j` in column `k` of row `i` means that the the cross action of the k^{th} simple root takes x_i to x_j .

The next n columns give Cayley transforms by noncompact imaginary roots. There is an entry in in column `k` of row `i` only if the k^{th} simple root is noncompact imaginary for x_i ; in this case an entry `j` means this Cayley transform takes x_i to x_j . These Cayley transforms are single valued (the inverse, multivalued Cayley transforms are not listed).

The final entry is the twisted involution in W corresponding to x_i , as a product of simple reflections (cf. Section 4).

Here is the $SL(2, \mathbb{R})$ example again:

```
0:  0  0  [n]  1  2
1:  0  0  [n]  0  2
2:  1  1  [r]  2  *  1
```

Here is the information we can read off from this output. Orbits #0 and #1 correspond to the compact Cartan subgroup (#0), and are 0-dimensional. Orbit #2 corresponds to the split Cartan subgroup and is one dimensional. The first two orbits are interchanged by the cross action of the simple root. For orbits #0,#1 this root is noncompact, and the Cayley transform takes each of these orbits to orbit #2.

See [1, Example 14.19] for a detailed discussion of $Sp(4, \mathbb{R})$.

Example 7.2 Here is $SL(3, \mathbb{R})$:

```
empty: kgb
Lie type: A2 sc s
there is a unique real form: sl(3,R)
kgbsize: 4
Name an output file (return for stdout, ? to abandon):
0:  0  0  [C,C]  2  1  *  *
1:  1  0  [n,C]  1  0  3  *  2,1
2:  1  0  [C,n]  0  2  *  3  1,2
3:  2  1  [r,r]  3  3  *  *  1,2,1
```

This is not an equal rank case; the dimension of the closed orbit is 1, and the dimension of the unique open orbit is $1 + 2 = 3$, the dimension of G/B .

Example 7.3 We illustrate one way to compute the order of \mathcal{X}^r using this information. Consider equal rank real forms of $SL(4, \mathbb{C})$, i.e. $SU(4)$, $SU(3, 1)$ and $SU(2, 2)$.

Using `kgb` we compute the number of orbits in each case, given by `kgbsize`. The result is:

G	$ K \backslash G/B $
$SU(2, 2)$	21
$SU(3, 1)$	10
$SU(4, 0)$	1

We will see later (Example 9.3) there are two strong real forms mapping to $SU(4)$ and $SU(3,1)$, and one mapping to $SU(2,2)$. This means that the set $K \backslash G/B$ for $SU(3,1)$, of order 10, appears twice in \mathcal{X}^r , and similiary for $SU(4)$. Therefore the order of \mathcal{X}^r is

$$(7.4) \quad 2 \times 1 + 2 \times 10 + 1 \times 21 = 43.$$

This agrees with the counting done a different way in Example 9.3.

Example 7.5 We do the same example again for equal rank forms of the adjoint group $PSL(4, \mathbb{C})$.

G	$ K \backslash G/B $
$PU(2,2)$	12
$PU(3,1)$	10
$PU(4,0)$	1

Since G is adjoint the map from strong real forms to real forms is bijective, so \mathcal{X}^r has order $12 + 10 + 1 = 23$. See Example 9.1

Example 7.6 If $G = Sp(4, \mathbb{R})$ there are 11 orbits of K on G/B , 4 of which are closed.

0:	0	0	[n,n]	1	2	6	4
1:	0	0	[n,n]	0	3	6	5
2:	0	0	[c,n]	2	0	*	4
3:	0	0	[c,n]	3	1	*	5
4:	1	2	[C,r]	8	4	*	* 2
5:	1	2	[C,r]	9	5	*	* 2
6:	1	1	[r,C]	6	7	*	* 1
7:	2	1	[n,C]	7	6	10	* 2,1,2
8:	2	2	[C,n]	4	9	*	10 1,2,1
9:	2	2	[C,n]	5	8	*	10 1,2,1
10:	3	3	[r,r]	10	10	*	* 1,2,1,2

For $G = PSp(4, \mathbb{R}) = SO(3, 2)$ there are 7 orbits, 2 of them closed:

0:	0	0	[n,n]	0	1	3	2
1:	0	0	[c,n]	1	0	*	2
2:	1	2	[C,r]	5	2	*	* 2
3:	1	1	[r,C]	3	4	*	* 1

4:	2	1	[n,C]	4	3	6	*	2,1,2
5:	2	2	[C,n]	2	5	*	6	1,2,1
6:	3	3	[r,r]	6	6	*	*	1,2,1,2

See Example 8.13.

Example 7.7 Complex Groups.

As in Example 6.10 let $G = G_1 \times G_1$, $K = G_1^\Delta$, and $G(\mathbb{R}) = G_1(\mathbb{C})$. Then $B = B_1 \times B_1$. It is well known, and easy to see, that $(g, h) \rightarrow gh^{-1}$ induces a bijection

$$(7.8) \quad G_1^\Delta \backslash G_1 \times G_1 / B_1 \times B_1 \simeq B_1 \backslash G_1 / B_1.$$

By the Bruhat decomposition the right hand side is parametrized by the Weyl group of G_1 .

For example take $G(\mathbb{R}) = G_1(\mathbb{C}) = SL(3, \mathbb{C})$.

```

main: type
Lie type: A2.A2 sc C
main: kgb
there is a unique real form: sl(3,C)
kgbsize: 6
Name an output file (hit return for stdout):
0:  2  1  2  1  * * * * [CCCC] 0
1:  4  0  3  0  * * * * [CCCC] 1 2,4
2:  0  3  0  4  * * * * [CCCC] 1 1,3
3:  5  2  1  5  * * * * [CCCC] 2 2,1,3,4
4:  1  5  5  2  * * * * [CCCC] 2 1,2,4,3
5:  3  4  4  3  * * * * [CCCC] 3 1,2,1,3,4,3

```

Since all roots are complex there are no Cayley transforms. The cross action of the Weyl group $W(A_2) \simeq S^3$ is simply transitive. The last column gives the Weyl group element (w, w) as a product of simple reflections; just taking the first half of each entry we see $W = \{id, s_2, s_1, s_2s_1, s_1s_2, s_1s_2s_1\}$. See Example 6.10.

8 Representation Theory: Blocks and the two-sided parameter space

The final ingredient in the algorithm is the *two-sided parameter space*. Let G^\vee be the dual group of G . The involution $\gamma \in \text{Out}(G)$ defines an involution $\gamma^\vee \in \text{Out}(G^\vee)$ and (G^\vee, γ^\vee) is also basic data. See [1, Section 2] for details. Let \mathcal{X}^\vee be the one-sided parameter space defined by (G^\vee, γ^\vee) .

Suppose $x \in \mathcal{X}$ and $\xi \in \tilde{\mathcal{X}}$ is a pre-image of x in $\tilde{\mathcal{X}}$. Then θ_ξ restricted to H is independent of the choice of ξ , and we denote it $\theta_{x,H}$. We also write $\theta_{x,H} \in \text{End}(\mathfrak{h})$ for its differential. There is a natural pairing between \mathfrak{h} and \mathfrak{h}^\vee (a Cartan subalgebra on the dual side); the adjoint $\theta_{x,H}^t$ of $\theta_{x,H}$ is an element of $\text{End}(\mathfrak{h}^\vee)$.

We can now define the *two-sided parameter space*:

$$(8.1) \quad \mathcal{Z} = \{(x, y) \in \mathcal{X} \times \mathcal{X}^\vee \mid \theta_{x,H}^t = -\theta_{y,H^\vee}\}.$$

See [1, Section 10]. We define the *reduced* two-sided parameter space \mathcal{Z}^r by replacing \mathcal{X} and \mathcal{X}^\vee with the corresponding reduced spaces (cf. Section 4). This is a finite set.

With the obvious notation $W \times W^\vee$ acts on \mathcal{Z} and \mathcal{Z}^r ; this is the cross action. By the condition relating x and y in (8.1), if α is a real root with respect to x then α^\vee is an imaginary root with respect to y , and vice-versa. Thus Cayley transforms are defined on \mathcal{Z}^r , via a real root on one side, and an imaginary noncompact root on the other.

The main result in [1], Theorem 10.3, (also see [1, Theorem 7.15]), is that the space \mathcal{Z} parametrizes the irreducible representations of strong real forms of G “up to translation”; equivalently with certain regular integral infinitesimal characters. Because of our emphasis on the reduced parameter space, we give a slightly different version here.

Apply the construction in the paragraph preceding (4.1) to choose a subset $Z^{\vee r}$ of the center Z^\vee of G^\vee . The pairing of \mathfrak{h} and \mathfrak{h}^\vee gives an isomorphism $\mathfrak{h}^* \simeq \mathfrak{h}^\vee$, and let

$$(8.2) \quad L = \{\lambda \in \mathfrak{h}^* \mid \exp(2\pi i\lambda) \in Z^{\vee r}\}.$$

Note that L is a subset of the weight lattice $P = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\}$. Choose a set Λ of representatives of $L/X^*(H)$, satisfying $\langle \lambda, \alpha^\vee \rangle \neq 0$ for all $\lambda \in \Lambda$ and all roots α . This is a finite set.

Choose a set $\{\xi_i \mid 1 \leq i \leq n\}$ as in (4.4). For each i let $G_i(\mathbb{R})$ be the real form of G defined by ξ_i , and let $\Pi(G_i(\mathbb{R}), \Lambda)$ be the set of irreducible admissible representations of $G_i(\mathbb{R})$ with infinitesimal character contained in Λ .

Theorem 8.3 *There is a natural bijection*

$$(8.4) \quad \mathcal{Z}^r \xleftrightarrow{1-1} \prod_{i=1}^n \Pi(G_i(\mathbb{R}), \Lambda).$$

This differs from [1, Theorem 10.3] in several ways. First of all on the left hand side we're using \mathcal{Z}^r instead of \mathcal{Z} . On the right hand side the union is over strong real forms in the sense of the reduced parameter space, instead of the possibly infinite set \mathcal{I}/G of [1, 5.15]. Finally the set $\Lambda = L/X^*(H)$ is contained in the corresponding set $P/X^*(H)$ of [1].

Fix $(x, y) \in \mathcal{Z}^r$, and consider the representations associated to the pairs (x', y') in the subset $\mathcal{X}^r[x] \times \mathcal{X}^{\vee, r}[y]$ of \mathcal{Z}^r . Choose i so that x is G -conjugate to x_i . Then these are representations of $G_i(\mathbb{R})$, all with the same infinitesimal character. In fact this set of irreducible representations is a *block* in the sense of [3, Chapter 9], and every block is obtained this way.

The entire construction is obviously symmetric in G and G^\vee , so an element $(x, y) \in \mathcal{Z}^r$, which defines a representation π of a real form of G , also defines a representation π^\vee of a real form of G^\vee . The map $\pi \rightarrow \pi^\vee$ is a version of *Vogan Duality*; see [4] and [1, 1.35 and Corollary 10.9]. In particular this is a duality of *blocks*: the set $\mathcal{X}^r[x] \times \mathcal{X}^{\vee, r}[y]$ defines a block \mathcal{B} of a real form of G , a block \mathcal{B}^\vee of a real form of G^\vee , and gives a bijection $\mathcal{B} \leftrightarrow \mathcal{B}^\vee$.

Before we look at individual representations, it is helpful to look at the sizes of blocks, which are given by the `blocksizes` command.

Example 8.5 Here are the blocks for real forms of $SL(2, \mathbb{C})$:

```
main: type
Lie type: A1 sc s
main: blocksizes
  0  1
  1  3
```

The rows and columns correspond to real forms of G and G^\vee , respectively. Adding some labelling by hand the picture is:

	$SO(3)$	$SO(2,1)$
$SU(2)$	0	1
$SL(2,\mathbb{R})$	1	3

Thus $SU(2)$ has a single block, dual to a block of $SO(2,1)$, and $SL(2,\mathbb{R})$ has two blocks, dual to $SO(3)$, and $SO(2,1)$, respectively. See Example 8.11.

Example 8.6 The corresponding output for $PSL(2,\mathbb{C})$ is the same, but there is a subtle point here.

```
main: type
Lie type: A1 sc s
main: blocksizes
  0    1
  1    3
```

Since the dual group $SL(2,\mathbb{C})$ is not adjoint, the map from strong real forms to real forms (on the dual side) is not injective. There are two strong real forms mapping to $SU(2)$, and we can label the strong real forms $SU(2,0)$, $SU(1,1)$ and $SU(0,2)$. Consequently there are two blocks of $SO(2,1)$ of size 1. See Example 8.12. We could display this information by constructing a table by hand showing *strong* real forms on the dual side:

	$SU(2,0)$	$SU(0,2)$	$SL(2,\mathbb{R})$
$SO(3)$	0	0	1
$SO(2,1)$	1	1	3

Blocks tend to be concentrated on the quasisplit forms of G or G^\vee .

Example 8.7 Here is the output of `blocksizes` for real forms of $Sp(12,\mathbb{C})$, with the real forms added:

	$SO(13)$	$SO(12,1)$	$SO(11,2)$	$SO(10,3)$	$SO(9,4)$	$SO(8,5)$	$SO(7,6)$
$Sp(6)$	0	0	0	0	0	0	1
$Sp(5,1)$	0	0	0	0	0	0	36
$Sp(4,2)$	0	0	0	0	0	0	315
$Sp(3,3)$	0	0	0	0	0	0	680
$Sp(12,\mathbb{R})$	1	13	108	556	1975	4707	7416

Example 8.8 In the case of equal rank real forms of $SO(12, \mathbb{C})$ we get some other elements:

	$SO(12)$	$SO(10, 2)$	$SO^*(12)$	$SO^*(12)$	$SO(8, 4)$	$SO(6, 6)$
$SO(12)$	0	0	0	0	0	1
$SO(10, 2)$	0	0	0	0	15	66
$SO^*(12)$	0	0	0	60	0	692
$SO^*(12)$	0	0	60	0	0	692
$SO(8, 4)$	0	15	0	0	300	885
$SO(6, 6)$	1	66	692	692	885	2320

Note that this diagram is symmetric: $SO(12, \mathbb{C})$ is self-dual (and the equal rank and split inner classes coincide). For a discussion of the two versions of $SO^*(12)$ see Example 3.3. Note that the two versions of $SO^*(12)$ on the dual side account for the fact that $SO(6, 6)$ has two distinct blocks of size 692.

Example 8.9 We break the symmetry between G and G^\vee of the previous example by taking $Spin(12, \mathbb{C})$, which is dual to $PSO(12, \mathbb{C})$.

	$PSO(12)$	$PSO(10, 2)$	$PSO^*(12)$	$PSO^*(12)$	$PSO(8, 4)$	$PSO(6, 6)$
$Spin(12)$	0	0	0	0	0	1
$Spin(10, 2)$	0	0	0	0	15	87
$Spin^*(12)$	0	0	0	60	0	692
$Spin^*(12)$	0	0	60	0	0	692
$Spin(8, 4)$	0	15	0	0	300	915
$Spin(6, 6)$	1	66	436	436	885	2180

There is an interesting phenomenon here. Note that $Spin(6, 6)$ has a block \mathcal{B}_{Spin} of size 66, dual to a block of $PSO(10, 2)$. From the previous example $SO(6, 6)$ also has a block of size 66, denoted \mathcal{B}_{SO} , dual to a block for $SO(10, 2)$. From the output of the `block` command one can see these blocks are isomorphic.

All representations in a block have the same central character, and the representations in \mathcal{B}_{Spin} factor to the image of $Spin(6, 6)$ in $SO(6, 6)$. The image of this map has index 2 in $SO(6, 6)$. Therefore it is not obvious that \mathcal{B}_{Spin} and \mathcal{B}_{SO} should be isomorphic (and this doesn't happen for the blocks of $Spin(6, 6)$ of size 436 and 2, 180).

The explanation is seen by looking at the dual side. The map $SO(10, 2) \rightarrow PSO(10, 2)$ is surjective. It follows that these dual blocks are isomorphic, hence the blocks themselves are isomorphic.

We leave it to the reader to see that the symmetry between the two versions of $SO^*(12)$ is broken by taking the “non-diagonal” quotient of $SO(12, \mathbb{C})$. See Examples 2.7 and 3.3.

Example 8.10 Here is the output of `blocksizes` for E_8 , with the real forms added:

	compact	quaternionic	split
compact	0	0	1
quaternionic	0	3150	73410
split	1	73410	453060

The quaternionic real form is the one denoted `e8(e7.su(2))` in the software. The software computes the structure of the block of size 453,060 easily. However computing Kazhdan-Lusztig-Vogan polynomials for this block required special techniques. See www.liegroups.org for more information.

We now look at individual representations in a few examples. The cases of real forms of $SL(2, \mathbb{C})$ and $PSL(2, \mathbb{C})$ are explained in detail in [1, Example 12.20]. In particular see the table at the end of Section 12. Here is a brief summary.

Example 8.11 $SL(2, \mathbb{R})$ and $SU(2)$:

```
empty: block
Lie type: A1 sc s
(weak) real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
possible (weak) dual real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 1
Name an output file (return for stdout, ? to abandon):
0(0,1):  0  0  [i1]  1  (2,*)
1(1,1):  0  0  [i1]  0  (2,*)
2(2,0):  1  1  [r1]  2  (0,1)  1
```

This is the block of $SL(2, \mathbb{R})$ of size 3 of example 8.5. We can take the infinitesimal character to be ρ , and this block contains the two discrete series representations #0, #1, and the trivial representation #2. The singleton block of $SL(2, \mathbb{R})$ is found taking the dual group to be compact:

```
possible (weak) dual real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 0
Name an output file (return for stdout, ? to abandon):
0(2,0):  1  1  [rn]  0  (*,*)  1
```

This is an irreducible (non-spherical) principal series representation of $SL(2, \mathbb{R})$ at ρ (with odd K -types).

Example 8.12 The adjoint group $PSL(2, \mathbb{R}) \simeq SO(2, 1)$ has a block of size 3 dual to the preceding one:

```
0(0,2):  0  0  [i2]  0  (1,2)
1(1,0):  1  1  [r2]  2  (0,*)  1
2(1,1):  1  1  [r2]  1  (0,*)  1
```

We can take infinitesimal character ρ , and the block consists of the unique discrete series representation #0 and the two one-dimensional representations #1, #2. These are all of the irreducible representations of $PSL(2, \mathbb{R})$ with infinitesimal character ρ .

As before taking the dual group to be compact we obtain a singleton:

```
possible (weak) dual real forms are:
0: su(2)
1: sl(2,R)
enter your choice: 0
Name an output file (return for stdout, ? to abandon):
0(1,0):  1  1  [rn]  0  (*,*)  1
```

This block occurs at infinitesimal character 2ρ ; in this case the set Λ of Theorem 8.3 can be taken to be $\{\rho, 2\rho\}$. Note that $SO(2, 1)$ has two irreducible representation PS_{\pm} at infinitesimal character 2ρ . The fact that there are two such representations (each of which is a block) is reflected in

the fact that on the dual side there are two strong real forms $SU(2, 0$ and $SU(0, 2)$. See Examples 8.5 and 8.6, and the table at the end of [1, Section 12].

We illustrate the other information in the output of `block` by looking at $Sp(4, \mathbb{R})$. See [1, Example 14.19].

Example 8.13 Here is the block of size 12 for $Sp(4, \mathbb{R})$:

```

0( 0,6):  0  0  [i1,i1]   1  2  ( 6, *)  ( 4, *)
1( 1,6):  0  0  [i1,i1]   0  3  ( 6, *)  ( 5, *)
2( 2,6):  0  0  [ic,i1]   2  0  ( *, *)  ( 4, *)
3( 3,6):  0  0  [ic,i1]   3  1  ( *, *)  ( 5, *)
4( 4,4):  1  2  [C+,r1]   8  4  ( *, *)  ( 0, 2)  2
5( 5,4):  1  2  [C+,r1]   9  5  ( *, *)  ( 1, 3)  2
6( 6,5):  1  1  [r1,C+]   6  7  ( 0, 1)  ( *, *)  1
7( 7,2):  2  1  [i2,C-]   7  6  (10,11) ( *, *)  2,1,2
8( 8,3):  2  2  [C-,i1]   4  9  ( *, *)  (10, *)  1,2,1
9( 9,3):  2  2  [C-,i1]   5  8  ( *, *)  (10, *)  1,2,1
10(10,0): 3  3  [r2,r1]  11 10  ( 7, *)  ( 8, 9)  1,2,1,2
11(10,1): 3  3  [r2,rn]  10 11  ( 7, *)  ( *, *)  1,2,1,2

```

The 12 representations, numbered $0, \dots, 11$, are parametrized by pairs (x, y) , the second entry on each line, from the corresponding `kgb` commands for G and G^\vee . See Example 7.6; note that for $G = Sp(4, \mathbb{R})$ there are 11 orbits numbered $0, \dots, 11$ and for $G^\vee = SO(3, 2)$ there are 7 orbits, labelled $0, \dots, 6$.

The next two columns give the Cartan subgroup, and length of the parameter, just as in the output of `kgb` for x (Section 7).

The term in brackets list each of the simple roots as:

- compact imaginary: `ic`
- noncompact imaginary, type I/II: `i1/i2`
- complex: `C+,C-`
- real, not satisfying the parity condition: `rn`
- real, satisfying the parity condition type I/II: `r1/r2`

For information about type I/II roots and the parity condition see [3, Section 8.3].

The next two columns give cross actions of the simple roots, similar to the `kgb` command, followed by two columns for Cayley transforms. These follow from the cross action/Cayley transforms for `kgb` on both the G and G^\vee side. In this setting (unlike `kgb`) these can be double valued, even for a noncompact imaginary root, since this corresponds to a real root on the dual side.

The final column is the corresponding twisted involution, exactly as in the `kgb` command for x .

Example 8.14 We conclude with a large example.

```
empty: block
Lie type: E6 sc c
(weak) real forms are:
0: e6
1: e6(so(10).u(1))
2: e6(su(6).su(2))
enter your choice: 2
possible (weak) dual real forms are:
0: e6(f4)
1: e6(R)
enter your choice: 0
```

This is the block for $E_6(A_5 \times A_1)$ (simply connected), dual to $E_6(F_4)$ (adjoint).

0(851,44):	8	4	[C+,rn,rn,rn,rn,C+]	2	0	0	0	0	1
1(1013,43):	9	4	[C+,rn,rn,rn,C+,C-]	4	1	1	1	3	0
2(1014,42):	9	4	[C-,rn,C+,rn,rn,C+]	0	2	5	2	2	4
3(1165,41):	10	4	[C+,rn,rn,C+,C-,rn]	8	3	3	6	1	3
4(1166,40):	10	4	[C-,rn,C+,rn,C+,C-]	1	4	9	4	8	2
5(1167,39):	10	4	[rn,rn,C-,C+,rn,C+]	5	5	2	7	5	9
6(1304,38):	11	4	[C+,C+,C+,C-,rn,rn]	11	10	10	3	6	6
7(1305,37):	11	4	[rn,C+,rn,C-,C+,C+]	7	14	7	5	14	12
8(1306,36):	11	4	[C-,rn,C+,C+,C-,rn]	3	8	13	11	4	8
9(1307,35):	11	4	[rn,rn,C-,C+,C+,C-]	9	9	4	12	13	5
10(1430,34):	12	4	[C+,C-,C-,rn,rn,rn]	18	6	6	10	10	10
11(1431,33):	12	4	[C-,C+,C+,C-,rn,rn]	6	18	15	8	11	11
12(1432,32):	12	4	[rn,C+,rn,C-,C+,C-]	12	19	12	9	17	7
13(1433,31):	12	4	[rn,rn,C-,C+,C-,rn]	13	13	8	16	9	13
14(1434,30):	12	4	[rn,C-,rn,rn,C-,C+]	14	7	14	14	7	19
15(1536,29):	13	4	[C+,C+,C-,C+,rn,rn]	24	24	11	20	15	15

16(1537,28):	13	4	[rn,C+,C+,C-,C+,rn]	16	22	20	13	21	16
17(1538,27):	13	4	[rn,C+,rn,C+,C-,C+]	17	23	17	21	12	23
18(1539,26):	13	4	[C-,C-,C+,rn,rn,rn]	10	11	24	18	18	18
19(1540,25):	13	4	[rn,C-,rn,rn,C+,C-]	19	12	19	19	23	14
20(1620,24):	14	4	[C+,C+,C-,C-,C+,rn]	28	26	16	15	25	20
21(1621,23):	14	4	[rn,C+,C+,C-,C-,C+]	21	29	25	17	16	27
22(1622,22):	14	4	[rn,C-,C+,rn,C+,rn]	22	16	26	22	29	22
23(1623,21):	14	4	[rn,C-,rn,C+,C-,C-]	23	17	23	27	19	17
24(1624,20):	14	4	[C-,C-,C-,C+,rn,rn]	15	15	18	28	24	24
25(1684,19):	15	4	[C+,C+,C-,rn,C-,C+]	32	31	21	25	20	30
26(1685,18):	15	4	[C+,C-,C-,C+,C+,rn]	34	20	22	34	31	26
27(1686,17):	15	4	[rn,C+,C+,C-,rn,C-]	27	33	30	23	27	21
28(1687,16):	15	4	[C-,C+,rn,C-,C+,rn]	20	34	28	24	32	28
29(1688,15):	15	4	[rn,C-,C+,C+,C-,C+]	29	21	31	33	22	33
30(1730,14):	16	4	[C+,C+,C-,rn,rn,C-]	37	36	27	30	30	25
31(1731,13):	16	4	[C+,C-,C-,C+,C-,C+]	38	25	29	35	26	36
32(1732,12):	16	4	[C-,C+,rn,rn,C-,C+]	25	38	32	32	28	37
33(1733,11):	16	4	[rn,C-,C+,C-,rn,C-]	33	27	36	29	33	29
34(1734,10):	16	4	[C-,C-,rn,C-,C+,rn]	26	28	34	26	38	34
35(1760, 9):	17	4	[C+,rn,C+,C-,C+,C+]	40	35	39	31	40	39
36(1761, 8):	17	4	[C+,C-,C-,C+,rn,C-]	41	30	33	39	36	31
37(1762, 7):	17	4	[C-,C+,rn,rn,rn,C-]	30	41	37	37	37	32
38(1763, 6):	17	4	[C-,C-,rn,C+,C-,C+]	31	32	38	40	34	41
39(1777, 5):	18	4	[C+,rn,C-,C-,C+,C-]	43	39	35	36	42	35
40(1778, 4):	18	4	[C-,rn,C+,C-,C-,C+]	35	40	42	38	35	43
41(1779, 3):	18	4	[C-,C-,rn,C+,rn,C-]	36	37	41	43	41	38
42(1786, 2):	19	4	[C+,rn,C-,rn,C-,C+]	44	42	40	42	39	44
43(1787, 1):	19	4	[C-,rn,C+,C-,C+,C-]	39	43	44	41	44	40
44(1790, 0):	20	4	[C-,rn,C-,rn,C-,C-]	42	44	43	44	43	42

The last column (twisted involution) has been deleted, as have the Cayley transforms, which are all empty, i.e. (*,*).

All of these representations come from a the same Cartan subgroup. This is the entry 4 in each row; by the output of the `cartan` command this is the most split Cartan subgroup, and is isomorphic to $\mathbb{R}^\times \times \mathbb{R}^\times \times \mathbb{C}^\times \times \mathbb{C}^\times$. Note that every root is of type C^\pm or `rn`, so there are no Cayley transforms. This is surprising, since this group has 5 conjugacy classes of Cartan subgroups including a compact Cartan subgroup but not a split one. The reason is that the dual block is for the adjoint group $E_6(F_4)$, which has only one conjugacy class of Cartan subgroup (cf. Example 5.4).

There are 1,791 orbits of K on G/B , and 45 for the dual group, given by the `kgb` command. In the parameters (x,y) given in parentheses each $0 \leq y \leq 44$ appears once, since the dual Cartan subgroup is connected. The fact that 45 distinct values of x (between 0 to 1790) occur is more subtle.

9 The Geometry of \mathcal{X}^r

Fix (G, γ) and let \mathcal{X}^r be the reduced parameter space defined in Section 4. It is helpful to give some detail about the structure of \mathcal{X}^r . Recall (Section 4) there is a map from \mathcal{X}^r to the space \mathcal{I}_W of twisted involutions in W . We discuss the fibers of this map.

9.1 The Adjoint Case

Fix (G, γ) with G adjoint. This case is simplest, since real forms and strong real forms coincide. Fix $\tau \in \mathcal{I}_W$.

Let $H^{-\tau} = \{h \in H \mid \tau(h) = h^{-1}\}$. There is a natural simply transitive action of $H^{-\tau}/(H^{-\tau})^0$ on \mathcal{X}_τ^r [1, Section 11]. Note that $H^{-\tau}/(H^{-\tau})^0 \simeq \mathbb{Z}_2^b$ where b is the number of S^1 factors of the real form of H defined by τ . Fix $x \in \mathcal{X}_\tau^r$. Via a choice of basepoint in \mathcal{X}_τ^r the output of `cartan` identifies \mathcal{X}_τ^r with $\{0, 1, \dots, 2^b - 1\}$. The element of \mathbb{Z}_2^b corresponding to $0 \leq k \leq 2^b - 1$ is the binary expansion of k .

The rank of \mathcal{X}_τ^r is also given by `fiber rank` in the output of `cartan`. For each Cartan the number `#X_r` is the number of strong involutions lying over this conjugacy class in \mathcal{I}_W , i.e. `twisted involution orbit size` \times $2^{\text{fiber rank}}$.

The imaginary Weyl group $W_{i,\tau}$ (the Weyl group of the imaginary roots with respect to τ) acts on \mathcal{X}_τ^r , and the orbits are in one to one correspondence with real forms containing this Cartan subgroup. The `cartan` command gives this decomposition. For example a line

```
real form #2: [0,1,2] (3)
```

means that corresponding to real form #2 (from the output of the `realform` command) are three elements of \mathcal{X}_τ^r , labelled 0,1,2 (the number in parentheses is the number of elements in brackets).

Example 9.1 Here are the equal rank real forms of $PSL(4, \mathbb{C})$:

```
empty: cartan
Lie type: A3 ad c
(weak) real forms are:
0: su(4)
1: su(3,1)
2: su(2,2)
```

enter your choice: 2
Name an output file (return for stdout, ? to abandon):

Cartan #0:
split: 0; compact: 3; complex: 0
canonical twisted involution:
twisted involution orbit size: 1; fiber rank: 3; #X_r: 8
imaginary root system: A3
real root system is empty
complex factor is empty
real form #2: [0,2,5] (3)
real form #1: [1,3,4,6] (4)
real form #0: [7] (1)

This is the fundamental fiber $\tau = \delta$. In this case $W_{i,\delta} = W$. This fiber has $2^3 = 8$ elements. There is a fixed point of the action of W on \mathcal{X}_δ , corresponding to the compact real form $PSU(4)$ (realform #0). The largest orbit, of size 4, corresponds to the group $SU(3,1)$.

Here are the remaining Cartan subgroups:

Cartan #1:
split: 0; compact: 1; complex: 1
canonical twisted involution: 1,2,3,2,1
twisted involution orbit size: 6; fiber rank: 1; #X_r: 12
imaginary root system: A1
real root system: A1
complex factor is empty
real form #2: [0] (1)
real form #1: [1] (1)

Cartan #2:
split: 1; compact: 0; complex: 1
canonical twisted involution: 2,1,3,2
twisted involution orbit size: 3; fiber rank: 0; #X_r: 3
imaginary root system is empty
real root system: A1.A1
complex factor: A1
real form #2: [0] (1)

There are 6 elements τ here for which \mathcal{X}_τ^r has order 2; both noncompact real forms appear. Finally only the quasisplit group $PSU(3, 3)$ contains the maximally split Cartan, in which case there are 3 twisted involutions τ , each with $|\mathcal{X}_\tau^r| = 1$.

Therefore \mathcal{X}^r has $1 \times 8 + 6 \times 2 + 3 \times 1 = 23$ elements. See Example 7.5

9.2 Strong Real Forms and Fibers

The `strongreal` command gives information about strong real forms (see Section 4), which may also be interpreted as information about the fibers $p: \mathcal{X}^r \rightarrow \mathcal{I}_W$ in the general case.

The strong real forms “containing” (H, τ) are parametrized by $\mathcal{X}_\tau^r/W_{i,\tau}$. In particular every strong real form contains the fundamental Cartan subgroup (H, δ) , so strong real forms are parametrized by $\mathcal{X}_\delta^r/W_{i,\delta}$. Things are particularly simple if G has equal rank, say n , in which case

$$(9.2) \quad \mathcal{X}_\delta^r/W \simeq \{h \in H \mid h^2 \in Z^r\}/W.$$

Note that $|\mathcal{X}_\delta^r| = |Z^r|2^n$ in this case.

In general not every element of Z^r is of the form x^2 for some $x \in \mathcal{X}$. The software labels the elements which are of this form as $\{z_0, \dots, z_{r-1}\}$, denoted `class #0`, `class #1`, `...`, `class #r-1`. In fact $r = 2^s$ for some s . For each z_j the set $\mathcal{X}_\tau^r(z_j) = \{x \in \mathcal{X}_\tau^r \mid x^2 = z_j\}$ is isomorphic to $H^{-\tau}/(H^{-\tau})^0$ (independent of j) which has cardinality 2^b (b is the number of S^1 factors as in the preceding section) so $|\mathcal{X}_\tau^r| = 2^{b+s}$.

The group $W_{i,\tau}$ acts on $\mathcal{X}_\tau^r(z_j)$, and the orbits correspond to strong real forms. All of this information is given by the `strongreal` command. See [1, Section 11] and the `help` file for the `strongreal` command.

Note that the order of \mathcal{X}^r may be computed by summing over Cartan subgroups, taking into account the size of the conjugacy classes of twisted involutions.

Example 9.3 Consider the equal rank inner class of $SL(4, \mathbb{C})$. Compare Example 9.1.

```
main: strongreal
Lie type: A3 sc c
(weak) real forms are:
0: su(4)
```

```

1: su(3,1)
2: su(2,2)
enter your choice: 0
there is a unique conjugacy class of Cartan subgroups
Name an output file (hit return for stdout):

```

there are 2 real form classes:

```

class #0:
real form #2: [0,1,2,4,5,6] (6)
real form #0: [3] (1)
real form #0: [7] (1)

```

```

class #1:
real form #1: [0,2,3,5] (4)
real form #1: [1,4,6,7] (4)

```

First of all, reading the lines beginning `realform`, we see there is one strong real form mapping to $SU(2, 2)$ (real form #2), two mapping to $SU(4)$ (real form #0), and two mapping to $SU(3, 1)$ (real form #1). We can think of these as $SU(4, 0)$, $SU(3, 1)$, $SU(2, 2)$, $SU(1, 3)$, and $SU(0, 4)$.

This is the fundamental fiber, i.e. $\tau = \delta$. In this case $Z \simeq \mathbb{Z}_4$ and $Z/Z^2 \simeq \mathbb{Z}_2$, so $|Z^r| = 2$. In this case b (the number of S^1 factors) is 3, so $|\mathcal{X}_\delta^r(z)| = 8$ for each $z \in Z^r$ (it is never 0 in this case), and $|\mathcal{X}_\delta^r| = 16$. We can take $Z/Z^2 = \{z_0, z_1\} = \{I, iI\}$. These are the elements `class #0` and `class #1` respectively.

There are three orbits of $W = W_{i,\delta}$ on $\mathcal{X}_\delta^r(I)$. The W -orbit of $\text{diag}(1, 1, -1, -1)$ has 6 elements; given in the line `real form #2: [0,1,2,4,5,6] (6)`; the corresponding real form is $SU(2, 2)$. The elements $\pm I$ are each fixed by W , and correspond to the two lines beginning `real form #0`. (The software does not specify which of these is I , and which $-I$).

We could have chosen $z_0 = -I$ instead of I . Then $SU(2, 2)$ would be given by i times the preceding ones, i.e. the W -orbit of $\text{diag}(i, i, -i, -i)$. Similarly the two strong real forms mapping to $SU(2)$ would be $\pm iI$. The software does not make an actual choice of Z^r ; the output of the software, and the combinatorics of the algorithm, are independent of any such choice. See [1, Section 13].

Now consider $z = iI$. Then $\mathcal{X}_\delta^r(iI)$ consists of the elements $\zeta(\pm 1, \dots, \pm 1)$

where $\zeta = e^{\pi i/4}$, with an odd number of plus signs. The 8 such elements constitute 2 W -orbits, hence the two strong real forms mapping to $SU(3, 1)$.

Here are the strong real forms containing the other two Cartan subgroups in this example.

```
real: strongreal
cartan class (one of 0,1,2): 1
Name an output file (return for stdout, ? to abandon):
```

there are 2 real form classes:

```
class #0:
real form #2: [0,1] (2)
```

```
class #1:
real form #1: [0] (1)
real form #1: [1] (1)
```

The compact real form **real form #0** only contains the compact Cartan subgroup and doesn't occur. In this case $|\mathcal{X}_r^r(I)| = |\mathcal{X}_r^r(iI)| = 2$, $|\mathcal{X}_r^r| = 4$, and both $SU(2, 2)$ and $SU(3, 1)$ occur.

```
real: strongreal
cartan class (one of 0,1,2): 2
Name an output file (return for stdout, ? to abandon):
```

```
real form #2: [0] (1)
```

For the most split Cartan subgroup $|\mathcal{X}_r^r(I)| = 0$, $|\mathcal{X}_r^r(iI)| = 1$, and only $SU(2, 2)$ occurs.

The number of twisted involutions in each conjugacy class are 1, 6 and 3, as given by the `cartan` command. Therefore the cardinality of \mathcal{X}^r is

$$(9.4) \quad 16 \times 1 + 4 \times 6 + 1 \times 3 = 43.$$

See Example 7.3.

Example 9.5 Type A_{2n} is a little different than A_{2n+1} . Here are the fundamental fiber and the equal rank strong real forms of $SL(5, \mathbb{C})$:

```

Lie type: A4 sc c
main: strongreal
(weak) real forms are:
0: su(5)
1: su(4,1)
2: su(3,2)
enter your choice: 0
there is a unique conjugacy class of Cartan subgroups
Name an output file (hit return for stdout):

```

```

real form #2: [0,1,2,4,5,6,8,10,11,13] (10)
real form #1: [3,9,12,14,15] (5)
real form #0: [7] (1)

```

In this case $Z \simeq \mathbb{Z}/5\mathbb{Z}$ and $Z/Z^2 = 1$. We can take $z = I$ in this case, and the map from strong real forms to real forms is bijective.

We conclude with a discussion of the inner class of $SL(n, \mathbb{R})$. If n is odd this is the unique real form in this inner class; if n is even there is one other real form $SL(n/2, \mathbb{H})$. We can take $\theta(\text{diag}(z_1, \dots, z_n)) = \text{diag}(\frac{1}{z_n}, \dots, \frac{1}{z_1})$, and $Z^r = \{I\}$ if n is odd, or $\{\pm I\}$ if n is even.

Here is an example of each case.

Example 9.6 There is only one strong real form in the inner class of $SL(3, \mathbb{R})$:

```

empty: strongreal
Lie type: A2 sc s
there is a unique real form: sl(3,R)

```

Here $Z^r = \{I\}$ and

$$(9.7) \quad \mathcal{X}_\delta(I) = \{\text{diag}(z, w, z) \mid z^2 w = 1\} / \{\text{diag}(ac, b^2, ac) \mid abc = 1\} = I$$

Example 9.8 Now consider the inner class of $SL(4, \mathbb{R})$, which also contains the real form $SL(2, \mathbb{H})$. In this case $Z = \pm I$.

```

empty: strongreal
Lie type: A5 sc s
(weak) real forms are:
0: sl(3,H)

```

```

1: sl(6,R)
enter your choice: 0
there is a unique conjugacy class of Cartan subgroups
Name an output file (return for stdout, ? to abandon):

```

there are 2 real form classes:

```

class #0:
real form #1: [0,1] (2)

```

```

class #1:
real form #0: [0] (1)
real form #0: [1] (1)

```

We compute

$$\begin{aligned}
H^{-\tau}/(H^{-\tau})^0 &= \{\text{diag}(z, w, w, z) \mid z^2 w^2 = 1\} / \{\text{diag}(ad, bc, bc, ad) \mid abcd = 1\} \\
&= \{I, \text{diag}(1, -1, -1, 1)\}.
\end{aligned}$$

Taking $z = I$ (this is **class #1**) we therefore have

$$(9.9) \quad \mathcal{X}_\delta(I) = \{\delta, \text{diag}(1, -1, -1, 1)\delta\}.$$

These elements are not conjugate by W , so there are two strong real forms mapping to the real form $SL(2, \mathbb{H})$ (**real form #0**).

On the other hand take $z = -I$ (**class #0**). Then $x = \text{diag}(1, 1, -1, -1)\delta \in \mathcal{X}_\delta(-I)$, and $\mathcal{X}_\delta(-I)$ is obtained by multiplying x on the left by $H^{-\tau}/(H^{-\tau})^0$. This gives

$$\mathcal{X}_\delta(-I) = \{\text{diag}(1, 1, -1, -1)\delta, \text{diag}(1, -1, 1, -1)\delta\}.$$

These elements are conjugate by W , so there is only one strong real form mapping to the real form $SL(4, \mathbb{R})$ (**real form #1**).

References

- [1] Jeffrey Adams and Fokko du Cloux. Algorithms for representation theory of real reductive groups, <http://www.liegroups.org/papers>. To appear in J. Inst. Math. Jussieu

- [2] Anthony Knapp. *Representation Theory of Semisimple Groups. An overview based on Examples*. Princeton University Press, Princeton, NJ, 1986.
- [3] David A. Vogan, Jr. *Representations of Real Reductive Lie Groups*, volume 15 of *Progress in mathematics*. Birkhäuser, Boston, 1981.
- [4] David A. Vogan, Jr. Irreducible characters of semisimple Lie groups IV. character-multiplicity duality. *Duke Math. J.*, 49, No. 4:943–1073, 1982.