

# The Atlas of Lie Groups and Representations

Jeffrey Adams

November 4, 2004

Atlas members:

Dan Barbasch

John Stembridge

Peter Trapa

David Vogan

Jiu-Kang Yu

`atlas.math.umd.edu`

For our purposes we may take a Lie group to be a group of (real or complex) matrices, such as the rotation group  $O(3)$ , the general linear group  $GL(n, \mathbb{R})$  (consisting of all  $n \times n$  invertible real matrices), an indefinite orthogonal group  $SO(p, q)$  (preserving a symmetric form of signature  $p, q$ ), or the symplectic group  $Sp(2n, \mathbb{R})$  (preserving a symplectic form).

A *representation* of  $G$  is a (continuous) homomorphism  $\pi : G \rightarrow \text{Aut}(V)$  of  $G$  to the (linear) automorphisms of a vector space. A representation  $\pi$  is *unitary* if  $V$  is a Hilbert space and the operators  $\pi(g)$  preserve the Hilbert space structure, i.e.

$$(\pi(g)v, \pi(g)w) = (v, w) \quad \text{for all } v, w \in V, g \in G$$

Lie groups and their representations are basic objects in mathematics, physics, and other sciences.

We say two representations  $\pi_1, \pi_2$  are *equivalent* if they are the same, up to a change of basis.

A unitary representation is *irreducible* if it cannot be written  $V = V_1 \oplus V_2$  where  $V_1, V_2$  are representations of  $G$ .

**Basic Problem:** Classify the set  $\widehat{G}$  of (equivalence classes of) irreducible unitary representations of any Lie group  $G$ .

**Note:** In general  $V$  is infinite dimensional, and some more explanation is needed to make the concepts of representation, unitary and irreducible precise.

**Example:** If  $G$  is a finite group then  $\widehat{G}$  is known by work of Frobenius and Schur. In particular  $|\widehat{G}|$  is the number of conjugacy classes of  $G$ .

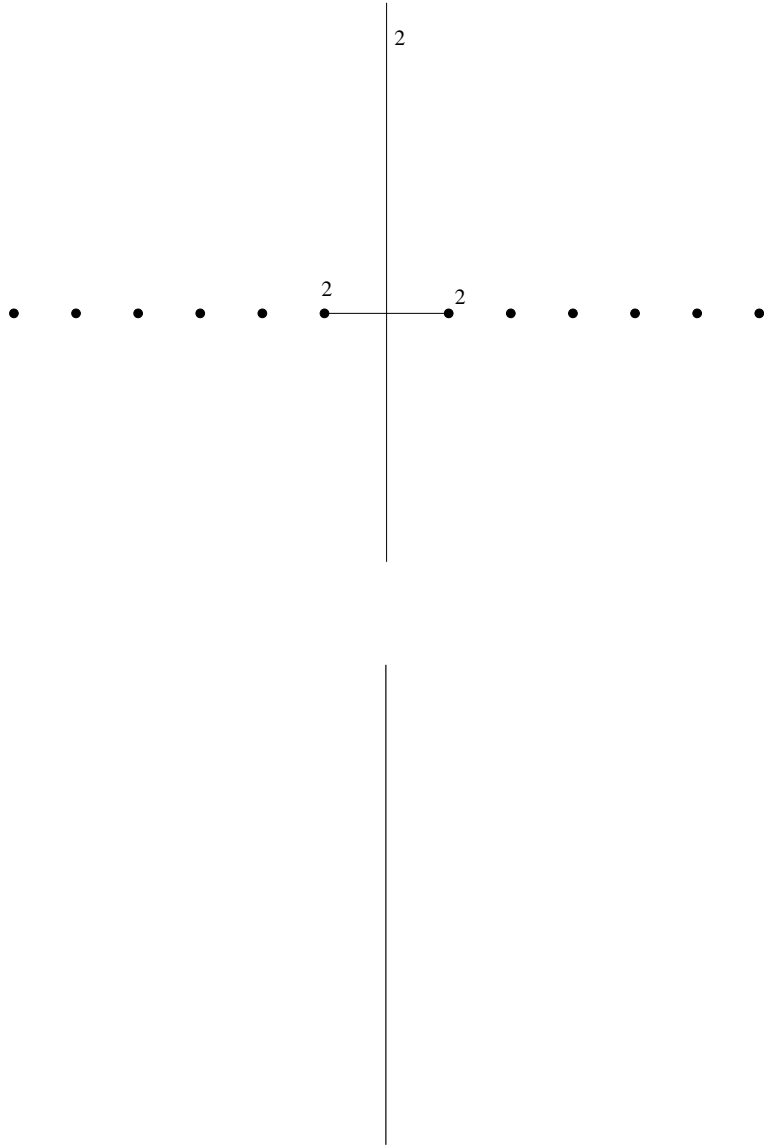
The **Atlas of Finite Groups** has a tremendous amount of information on finite groups and their representations.

**Example:** If  $G = SO(3)$ , the  $3 \times 3$  matrices of determinant 1, satisfying  $gg^t = I$ , then there is one irreducible representation of  $G$  of dimension  $n$  for all  $n > 0$  odd.

**Example:** If  $G$  is a compact group (like  $SO(3)$ ) then every irreducible representation of  $G$  is unitary and finite dimensional, and they were

classified by Cartan and Weyl in the 1920s. They are parametrized by a lattice intersected with a cone in  $\mathbb{R}^n$ .

**Example:** Take  $G = SL(2, \mathbb{R})$ , the  $2 \times 2$  real matrices of determinant 1. Then  $\widehat{G}$  was computed by Bargmann in the 1940s. Here is a picture:



The Problem of the Unitary Dual has been the object of intensive study for over 50 years. For non-compact groups it is only known in a few cases, such as  $SL(2, \mathbb{R})$  and  $GL(n, \mathbb{R})$ .

Let  $G$  be any given Lie group.

**Theorem 0.1 (Vogan)** There is a finite algorithm to compute the unitary dual of  $G$ .

**Note:** This is much weaker than saying there is a finite algorithm for a single family of groups such as  $GL(n, \mathbb{R})$ .

**Proposal A:** Write a computer program to compute the unitary dual of a fixed real group.

**Note:** There is a big difference between a theorem asserting the existence of a finite algorithm, and a computer program.

**Proposal B:** Create a web site to make information about Lie groups and their representations available to the mathematical and scientific communities - much like the Atlas of Finite Groups.

## Why do this?

- To tell us more about the unitary dual.

## Why do this?

- To tell us more about the unitary dual.
- As a way to learn about the unitary dual and the technology involved.

## Why do this?

- To tell us more about the unitary dual.
- As a way to learn about the unitary dual and the technology involved.
- Examples are fascinating; computers make more complicated examples accessible.

## Why do this?

- To tell us more about the unitary dual.
- As a way to learn about the unitary dual and the technology involved.
- Examples are fascinating; computers make more complicated examples accessible.
- There are many interesting partial problems to be solved. It is not necessary to compute the full unitary dual to be successful.

## Why do this?

- To tell us more about the unitary dual.
- As a way to learn about the unitary dual and the technology involved.
- Examples are fascinating; computers make more complicated examples accessible.
- There are many interesting partial problems to be solved. It is not necessary to compute the full unitary dual to be successful.
- It is interesting and fun. A lot of people will be involved (I hope).

## Why do this?

- To tell us more about the unitary dual.
- As a way to learn about the unitary dual and the technology involved.
- Examples are fascinating; computers make more complicated examples accessible.
- There are many interesting partial problems to be solved. It is not necessary to compute the full unitary dual to be successful.
- It is interesting and fun. A lot of people will be involved (I hope).
- The Atlas Web Site will be an invaluable tool to experts, non-experts, and students.

It is difficult to compute examples even for the most proficient expert, and for non-experts it is almost impossible.

## Why not to do this?

- Maybe we won't understand about the unitary dual, even if we get the answer.

## Why not to do this?

- Maybe we won't understand about the unitary dual, even if we get the answer.
- Poor substitute for thinking.

## Why not to do this?

- Maybe we won't understand about the unitary dual, even if we get the answer.
- Poor substitute for thinking.
- Generate lots of data which is hard to interpret.

## Why not to do this?

- Maybe we won't understand about the unitary dual, even if we get the answer.
- Poor substitute for thinking.
- Generate lots of data which is hard to interpret.
- Huge investment of time.

## Why not to do this?

- Maybe we won't understand about the unitary dual, even if we get the answer.
- Poor substitute for thinking.
- Generate lots of data which is hard to interpret.
- Huge investment of time.
- Likelihood of failure: it is not clear this can be done.

## Why not to do this?

- Maybe we won't understand about the unitary dual, even if we get the answer.
- Poor substitute for thinking.
- Generate lots of data which is hard to interpret.
- Huge investment of time.
- Likelihood of failure: it is not clear this can be done.
- Can't trust computers. How do you know if the answer is correct?

Examples:

- Atlas of Finite Groups (mistake in character tables, which was caught)

Examples:

- Atlas of Finite Groups (mistake in character tables, which was caught)
- Maple test for positive semidefinite (was wrong in Maple 4)

Examples:

- Atlas of Finite Groups (mistake in character tables, which was caught)
- Maple test for positive semidefinite (was wrong in Maple 4)
- $\int \sin^{10}(x) \cos(x) dx =$

Examples:

- Atlas of Finite Groups (mistake in character tables, which was caught)
- Maple test for positive semidefinite (was wrong in Maple 4)

- $\int \sin^{10}(x) \cos(x) dx =$

$$\frac{21}{512} \sin(x) - \frac{15}{512} \sin(3x) + \frac{15}{512} \sin(35x) - \frac{5}{1024} \sin(7x) + \frac{11}{11264} \sin(9x) + C$$

- Positive definiteness revisited:

Let

$$X := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 7 \end{pmatrix}$$

I would like to know if this matrix is positive definite or not. Mathematica returns an error when I try this.

- Positive definiteness revisited:

Let

$$X := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 7 \end{pmatrix}$$

I would like to know if this matrix is positive definite or not. Mathematica returns an error when I try this.

According to Mathematica the eigenvalues are:

$$\begin{aligned} & \frac{11}{3} + \frac{23 \cdot 5^{\frac{2}{3}}}{3 (241 + 9i\sqrt{34})^{\frac{1}{3}}} + \frac{(5 (241 + 9i\sqrt{34}))^{\frac{1}{3}}}{3} \\ & \frac{11}{3} - \frac{23 \cdot 5^{\frac{2}{3}} (1 + i\sqrt{3})}{6 (241 + 9i\sqrt{34})^{\frac{1}{3}}} - \frac{(1 - i\sqrt{3}) (5 (241 + 9i\sqrt{34}))^{\frac{1}{3}}}{6} \\ & \frac{11}{3} - \frac{23 \cdot 5^{\frac{2}{3}} (1 - i\sqrt{3})}{6 (241 + 9i\sqrt{34})^{\frac{1}{3}}} - \frac{(1 + i\sqrt{3}) (5 (241 + 9i\sqrt{34}))^{\frac{1}{3}}}{6} \end{aligned}$$

- Positive definiteness revisited:

Let

$$X := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 7 \end{pmatrix}$$

I would like to know if this matrix is positive definite or not. Mathematica returns an error when I try this.

According to Mathematica the eigenvalues are:

$$\begin{aligned} & \frac{11}{3} + \frac{23 \cdot 5^{\frac{2}{3}}}{3 (241 + 9i\sqrt{34})^{\frac{1}{3}}} + \frac{(5 (241 + 9i\sqrt{34}))^{\frac{1}{3}}}{3} \\ & \frac{11}{3} - \frac{23 \cdot 5^{\frac{2}{3}} (1 + i\sqrt{3})}{6 (241 + 9i\sqrt{34})^{\frac{1}{3}}} - \frac{(1 - i\sqrt{3}) (5 (241 + 9i\sqrt{34}))^{\frac{1}{3}}}{6} \\ & \frac{11}{3} - \frac{23 \cdot 5^{\frac{2}{3}} (1 - i\sqrt{3})}{6 (241 + 9i\sqrt{34})^{\frac{1}{3}}} - \frac{(1 + i\sqrt{3}) (5 (241 + 9i\sqrt{34}))^{\frac{1}{3}}}{6} \end{aligned}$$

Numerically:

$$\begin{aligned} & 10.79 + 0.i \\ & -0.34 + 4.44 \times 10^{-16}i \\ & 0.54 - 4.44 \times 10^{-16}i \end{aligned}$$

## Some Computational Issues

### 1. Models of Representations of Weyl Groups

Suppose  $W$  is a Weyl group. This is a finite group, generated by reflections, associated to a root system  $R$  of type  $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4$  or  $G_2$ .

Type	$ R $	Order(W)	realization
$A_n$	$\frac{n(n-1)}{2}$	$n!$	$S^n$
$B_n/C_n$	$2n^2$	$2^n n!$	$S^n \times \mathbb{Z}_2^n$
$D_n$	$2n^2 - 2n$	$2^{n-1} n!$	$S^n \times \mathbb{Z}_2^{n-1}$
$E_6$	72	51,840	$O(6, \mathbb{F}_2)$
$E_7$	126	2,903,040	$O(7, \mathbb{F}_2) \times \mathbb{Z}/2\mathbb{Z}$
$E_8$	240	696,729,600	$W \xrightarrow{2} O(8, \mathbb{F}_2)$
$F_4$	48	1152	
$G_2$	12	12	$D_6$

Representations of Weyl groups:

Type	Number	Maximum Dimension
$A_n$	$n!$	?
$B_n/C_n$	?	?
$D_n$	?	?
$E_6$	25	90
$E_7$	60	512
$E_8$	112	7,168
$F_4$	25	16
$G_2$	6	2

The irreducible representations of  $W$  are classified (the character table is known).

**Problem:** Write down explicit matrices realizing the irreducible representations of a Weyl group.

That is  $W$  is generated by elements  $g_1, \dots, g_n$ , with relations (for example  $g_i^2 = 1$ ). For a given representation  $\pi$  write down matrices  $A_1, \dots, A_n$  so that  $\pi(g_i) = A_i$ .

This is a hard problem, and there is no general solution for a general finite group.

We have done this over the rationals for all exceptional groups, and over the integers for all but  $E_8$ . One method is to use the “meataxe” algorithm from finite group theory.

An issue which arises is: suppose  $M$  is an  $m \times n$  matrix with integral entries. View this as a homomorphism from  $\mathbb{Z}^n$  to  $\mathbb{Z}^m$ . Find a basis of the kernel of  $M$ . Equivalently: find a basis of the row space of  $M$  (over  $\mathbb{Z}$ !).

The solution is given by Smith Normal Form. Implementing it runs into issues of “integer ex-

plosion”.

We have many problems of this type involving **integral** or **rational** linear algebra.



















## 2. Testing for Positive Semidefiniteness

Suppose  $A$  is an  $n \times n$  symmetric matrix with rational entries. How do you tell if  $A$  is positive semi-definite? That is we can find  $B$  so that  $BAB^t$  is a diagonal matrix, with real entries. Are all of the entries non-negative? Equivalently are all eigenvalues of  $A$  (which are necessarily real) non-negative?

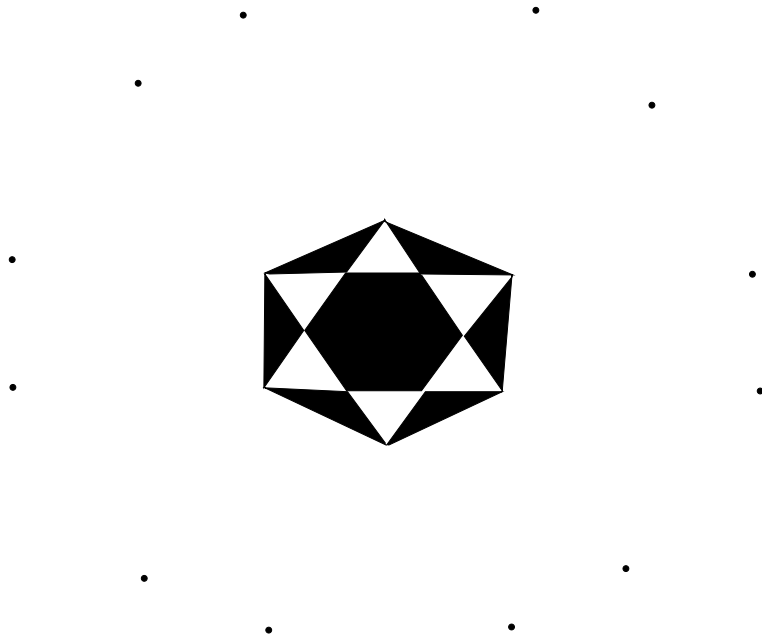
The standard test is: if and only if every minor has non-negative determinant. There are  $2^n$  such minors. If  $n = 7, 168$  this won't do; there are better methods, but we are not sure if they will suffice.

Note that we need to do rational, and not real, arithmetic. An eigenvalue may be a very small real number; to tell whether it is positive or negative it is not sufficient to work with any fixed precision. (The *sgn* function is very difficult to compute).

How to present the answer?

The unitary dual of  $G$  is a countable union of sets, each of which is the quotient of a subset of a vector space by a finite group (something like the Weyl group).

The spherical unitary dual of  $G_2$  (this is a subset of one of the vector spaces  $V$ ):



The subset of each vector space is a finite union of facets. We can give a representative point on facet. This gives a list, which may or may not be useful. We need to make the answer comprehensible (by first comprehending it!).