The Atlas of Lie Groups

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Let G be a real reductive group.

Theorem 0.1 (Vogan) There is a finite algorithm to compute the unitary dual of G.

Note: This is much weaker than saying there is a finite algorithm for a single family of groups such as $SL(n)$ or $SO(n,\mathbb{C})$.

Proposal: Write a computer program to compute the unitary dual of a fixed real group.

Note: There is a big difference between a theorem asserting the existence of a finite algorithm, and a computer program.

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The Atlas of Lie Groups has a second branch:

Proposal: Create a web site to make information about Lie groups and their representations available to the mathematical and scientific communities - much like the Atlas of Finite Groups.

(More on this later.)

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- Examples are fascinating; computers make more complicated examples accessible.
- There are many interesting partial problems to be solved. It is not necessary to compute the full unitary dual to be successful.
- It is fun. A lot of people will be involved (I hope).

Best outcome: based on what we learn from the computer we can compute the answer ”by hand”.

Example:

- Model orbit in E8 (Adams, Huang, Vogan),
- Non-arithmetic groups (Mostow, Deligne)

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- Generate lots of data which is hard to interpret.
- Huge investment of time.
- Likelihood of failure: it is not clear this can be done.
- Can't trust computers. How do you know if the answer is correct?

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$$\frac{21}{512}\sin(x) - \frac{15}{512}\sin(3x) + \frac{15}{512}\sin(35x)$$
$$-\frac{5}{1024}\sin(7x) + \frac{11}{11264}\sin(9x) + C$$

Sketch of proof of the Theorem:

Uses all of the machinery of representation theory of real groups. In particular:

Standard module: $I(P, \sigma) = Ind_P^G(\sigma)$, with irreducible quotient $\overline{I}(P, \sigma)$

Derived functor module: $R_{\mathfrak{q}}(\sigma)$

The notions of real infinitesimal character, Hermitian representation, K–character…

Theorem 0.2 (Knapp/Vogan) Suppose π is an irreducible unitary representation of G. Then $\pi = I(P, \sigma)$ for some irreducible unitary representation σ with real infinitesimal character (readily computable).

Definition 0.3 (Salamanca,Vogan) An infinitesimal character is small if it is real and in the convex hull of $W\rho$. An admissible representation is small if it has small infinitesimal character.

Conjecture 0.4 (Salamanca,Vogan) Suppose π is an irreducible unitary representation with real infinitesimal character. Then $\pi = Rq(\sigma)$ where σ is small $(\mathfrak{q}, \sigma)$ are readily computable).

Proposition 0.5 Proposition: There is a finite algorithm to compute the small unitary representations.

By the Conjecture it is enough to compute the small unitary representations, and by the

Proposition there is a finite algorithm to do this:

Corollary 0.6 Assuming the conjecture, there is a finite algorithm to compute the unitary dual.

Remark 0.7 In the absence of the conjecture there is still a finite algorithm, but it is probably useless from a computational point of view.

Sketch of the proof of the Proposition

Basic point:

- There is a finite number of minimal K-types μ to consider,
- For each μ a finite number of representations π of G,
- For each π a finite number of K-types to check for unitarity.

Definition 0.8 A K-type is unitarily small if its highest weight is in the convex hull of the weights of the exterior algebra of the weights of $g/\mathfrak{k}$.

Fix a K-type μ. Associated to μ is a parabolic subgroup $P = MAN$ and a representation π of M such that every irreducible representation with this lowest K-type is the unique irreducible quotient $I(P, \pi \otimes \nu)$ of $I(P, \pi \otimes \nu)$ for some ν.

Lemma 0.9 Unitarity of $I(P, \mu \otimes \nu)$ is constant on "reducibility" regions in the ν plane. There is only a finite number of such regions (with small infinitesimal character).

Summary:

- Reduce to small real infinitesimal character,
- Fix a unitarily small K-type,
- Compute finite number of regions in the ν plane,
- Pick a representative point in each such region. Compute the signature of the Hermitian form on every unitarily small K-type.

Computational Ingredients

- Encoding of all of the objects: Cartan subgroups, standard modules, Weyl groups;
- Multiplicity of K-types in standard modules – "easy", but includes branching laws for disconnected compact groups, for example from $O(n)$ to $O(k_1) \times O(k_r)$;
- Kazhdan–Lusztig polynomials, at all infinitesimal characters;
- Computation of Hermitian form on all unitarily small K-types;
- Models of representations of Weyl groups.

Example/Test Case:

Spherical Unitary Representations

Let G be a split group over a real or p–adic field $\mathbb{F}$. There are two ways to compute the spherical unitary dual of G: pure thought, and by computer.

Theorem 0.10 (Barbasch,Vogan) Let G be a classical group over $\mathbb{F}$ [Barbasch, Vogan $GL(n)$] or G_2 over $\mathbb{R}$ [Vogan]. Then the spherical unitary dual of $\mathbb{G}$ is known explicitly. In the classical case the answer is independent of $\mathbb{F}$.

Theorem 0.11 (Adams, Stembridge, Yu) Let G be a split p–adic reductive group over a p–adic field of type F_4 or E_6. Then the spherical unitary dual is known by computer calculation.

Barbasch has independently done F_4. Soon E_7 will be done also, but it is not clear E_8 can be done this way. We have also checked a number of classical cases and G_2 by computer.

It is *very difficult* to compare Barbasch/Vogan's answer with ours.

Spherical Unitary Explorer

Barbasch's result for classical groups is very beautiful. The answer has a simple form (which also holds for G_2 and presumably in the exceptional cases as well). However it is difficult, even for the cognoscenti, to understand his result.

As a demonstration of the second branch of the Atlas, making results known to the community, we have created the Spherical Unitary Explorer (currently for $Sp(2n)$).

Here is a sketch of Barbasch's result.

Fix G a split classical group, and let ${}^\vee G$ be the dual group. According to Arthur's conjectures, associated to a nilpotent orbit ${}^\vee O \subset {}^\vee G$ is a unipotent unitary spherical representation $\overline{I}(\nu)$ with $\nu = \frac{1}{2}h(\mathcal{O})$. All spherical unitary representations should be obtained by "deformation" from these.

Fix ν. Barbasch gives an algorithm which attaches to ν a nilpotent orbit ${}^\vee O$ with the following properties. Associated to ${}^\vee O$ is a Levi factor $M \subset G$ and the reductive part ${}^\vee Z$ of the centralizer of ${}^\vee O$. Then we may write

$$\nu = \frac{1}{2}{}^\vee\mathcal{O} + \nu'$$

where ν' is central in M. Furthermore ν corresponds to a representation $\overline{I}_Z(\nu)$. The algorithm, when applied to Z gives the 0–orbit.

Then $\overline{I}_G(\nu)$ is unitary if and only if $\overline{I}_Z(\nu)$ is unitary.

This reduces the problem to the computation of the unitary spherical principal series associated to the 0–orbit, which is computed by hand.

Example: $Sp(4)$

$\mathcal{O}$	ν	M	$^\vee Z$
(5)	$(2,1)$	$Sp(4)$	$O(1)$
$(3,1,1)$	$(1,0)$	$SL(2)\times GL(1)$	$O(2)$
$(2,2,1)$	$(1/2,1/2)$	$GL(2)$	$SL(2)$
$(1,1,1,1,1)$	$(0,0)$	$GL(1)\times GL(1)$	$SO(5)$