Computer Calculations in the Unitary Dual

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Abstract

These are notes on computing the unitary dual of a real Lie group by computer. They are based on lectures by David Vogan at the conference "Workshop on Computational Lie Theory" at CRM, Montreal, June 2002.

1 Introduction

1.1 Notation

Let G be a real reductive group. There is some flexibility in the precise hypotheses on G. Certainly we want to allow G to be disconnected, but probably want to assume it is in Harish-Chandra's class. Certainly taking G to be the real points of a connected complex reductive group would suffice. The identity component of such a group would also be allowed. We can allow G to be non–linear for many statements, but presumably not all.

To simplify statements it is helpful to assume that G is semi–simple, which we do from now on.

We write $\mathfrak{g}_0 = Lie(G)$ and $\mathfrak{g} = Lie(G) \otimes \mathbb{C}$. We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, with associated Cartan involution θ.

1.2 Building Blocks

There are two basic constructions of unitary representations. See [3] and [1].

Real parabolic induction:

$$I = Ind_P^G(\sigma \otimes e^{i\nu} \otimes 1) \tag{1}$$

Here $P = MAN$, σ is an irreducible unitary representation of M with real infinitesimal character ([3], page 535), $\nu \in i\mathfrak{a}_0^*$, and ν is regular.

Cohomological Induction:

$$R_{\mathfrak{q}}(\sigma) \tag{2}$$

Here $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is a θ–stable parabolic subalgebra of $\mathfrak{g}$, and σ is a unitary representation of L, with real infinitesimal character, in the good range ([1], Definition 0.49).

Definition 1.1 *We say an irreducible unitary representation π of G is* small *if the infinitesimal character λ of π is real, and contained in the convex hull of $W\rho$.*

Let $\hat{G}_{small}$ be the set of small irreducible unitary representations of G.

Proposition 1.2 ([3], Theorem 16.10) *Suppose π is an irreducible unitary representation of G. Then $\pi = Ind_P^G(\sigma)$ where σ is an irreducible unitary representation of M with real infinitesimal character.*

Conjecture 1.3 (Salamanca, Vogan) *Suppose π is an irreducible unitary representation of G with real infinitesimal character. Then $\pi = R_{\mathfrak{q}}(\sigma)$ where σ is a small unitary representation of L in the good range.*

The conjecture, together with Proposition 1.2 reduces the computation of the unitary dual to that of the small unitary representations of G and of each of its theta–stable Levi subgroups L.

From now on we concentrate on the classification of $\hat{G}_{small}$.

Proposition 1.4 *There is a finite algorithm to compute $\hat{G}_{small}$.*

This will take most of the remainder of these notes.

Corollary 1.5 *Assuming Conjecture 1.3 for G there is a finite algorithm to compute the unitary dual of G.*

In the absence of the conjecture there is still a finite algorithm to compute $\hat{G}$:

Lemma 1.6 *There is bounded region S of the space of real infinitesimal characters such that the following holds. Suppose π is an irreducible unitary representation of G with real infinitesimal character. Then $\pi = R_{\mathfrak{q}}(\sigma)$ where σ is a unitary representation of L, in the good range, with real infinitesimal character $\lambda \in S$.*

For simplicity we can take S to be the set of infinitesimal characters in the convex hull of $n\rho$ for some n. The conjecture says we can take $n = 1$. If we let $\hat{G}_{n-small}$ be the representations satisfying this condition we are reduced to computing $\hat{G}_{n-small}$. The arguments to prove Proposition 1.4 apply equally well to compute $\hat{G}_{n-small}$, although the size of the computation will increase dramatically.

2 Unitarity and K–types

Suppose X is a Harish–Chandra module with an invariant Hermitian form. If V is an irreducible K–submodule of X then the Hermitian form restricted to V is a multiple of the unique K–invariant Hermitian form on V. The Hermitian form on V is determined by this scalar.

For $\mu \in \hat{K}$ we write $X(\mu)$ for the μ–isotypic subspace of X. Then $X(\mu) \simeq \mu \otimes Hom_K(\mu, X)$. The Hermitian form restricted to $X(\mu)$ determines a Hermitian form on $Hom_K(\mu, X)$, and vice versa. By the restriction of the form to $X(\mu)$ we mean the associated Hermitian form on $Hom_K(\mu, X)$.

3 Unitarily small representations of K

Definition 3.1 *A representation μ of K is said to be* unitarily small *if the heighest weight (equivalently all weights) of μ is contained in the convex hull of the weights of $\Lambda^*(\mathfrak{g}/\mathfrak{k})$.*

Let $\hat{K}_{us}$ be the set of unitarily small K–types.

Proposition 3.2 *Suppose X is an admissible representation of G with real infinitesimal character contained in the convex hull of $W\rho$. Then*

1. *The lowest K–type of X is contained in $\hat{K}_{us}$.*
2. *Suppose X is Hermitian. If the Hermitian form is positive definite on $X(\mu)$ for all $\mu \in \hat{K}_{us}$ then X is unitary.*

4 Computing unitary representations with a given unitarily small K–type

Fix $\mu \in \hat{K}_{us}$ (Definition 3.1). By [6] associated to μ is a real parabolic subgroup $P = MAN \subset G$ and an irreducible unitary representation σ of M. For $\nu \in \mathfrak{a}^*$ let

$$X(\nu) = Ind_P^G(\sigma \otimes \nu \otimes 1)$$

This has real infinitesimal character if and only if $\nu \in \mathfrak{a}_0^*$. Write λ_M for the infinitesimal character of σ. Then the infinitesimal character λ of $X(\nu)$ may be written $\lambda = (\lambda_M, \nu)$, and $X(\nu)$ is irreducible unless (possibly) $< \lambda, \alpha\check{} > \in \mathbb{Z}$ for some root α satisfying $\alpha|_{\mathfrak{a}_0} \neq 0$.

We can regard the $(\mathfrak{g}, K)$–modules $X(\nu)$ as all being defined on the same vector space with the same action of K. In particular for $\mu \in \hat{K}$ we may identify the spaces $Hom_K(\mu, X(\nu))$ for all ν. Each representation $X(\nu)$ contains the K–type μ as a lowest K–type with multiplicity one.

Lemma 4.1 *Every irreducible representation π of G with μ as a lowest K-type is isomorphic to the unique irreducible summand $\overline{X}(\nu)$ containing μ for some $\nu \in \mathfrak{a}^*$. The infinitesimal character of $X(\nu)$ is real if and only if $\nu \in \mathfrak{a}_0^*$.*

We are interested in representations with real infinitesimal character so we will assume $\nu \in \mathfrak{a}_0^*$ from now on.

Let $W(\sigma)$ be the stabilizer of σ in $N_G(A)/MA$. Then $X(\nu)$ has an invariant Hermitian form if and only if there is $w_0 \in W(\sigma)$ satisfying $w_0(\nu) = -\nu$.

So we consider $\nu \in \mathfrak{a}_0^*$, contained in the convex hull of $W\rho$, such that $w_0\nu = -\nu$ for some w_0. Here is an algorithm for computing the unitarity of $\overline{X}(\nu)$, for any such ν.

1. For every $\mu' \in \hat{K}_{us}$, calculate the multiplicity $\overline{m}(\mu', \nu)$ of μ' in $\overline{X}(\nu)$.

2. Choose a vector v in the K–type μ. Choose a basis $\phi_1, \dots, \phi_{m(\mu')}$ of $\mathrm{Hom}_K(\mu', X(\nu))$, and write $v_i = \phi_i(v)$. Thus $\{v_1, \dots, v_{m(\mu')}\}$ generate $X(\nu)(\mu')$ as a representation of K, independent of ν.

3. For all $1 \leq i, j \leq N$ compute the inner product $< v_i, v_j > (\nu)$ of v_i, v_j in $X(\nu)$. This gives the $N \times N$ matrix, which we denote $A_{\mu'}(\nu)$, of the form on $\mathrm{Hom}_K(\mu', X(\nu))$ with respect to the basis $\{\phi_i\}$.

Notes:

- For 1. we need to compute the multiplicity of μ' in all standard modules. This is a non–trivial problem even for the minimal principal series of a split group.

- For 1. we also need to use the Kazhdan–Lusztig algorithm (at general infinitesimal character) to write $\overline{X}(\nu)$ as a linear combination of standard modules.

- We may take each v_i to be highest weight vector of the $m(\mu')$ irreducible summands of $X(\nu)$. This may not be the most useful choice.

- The matrix $A_{\mu'}(\nu)$ has rank $\overline{m}(\mu', \nu) \leq m(\mu')$, with equality for almost all ν. We may assume $< \nu, \alpha\check{} > \in \mathbb{Q}$ for all roots α. For appropriate choice of v and $\{\phi_i\}$ the coordinates of $A_{\mu'}$ are rational functions of ν.

Lemma 4.2 *$\overline{X}(\nu)$ is unitary if and only if $M_{\mu'}(\nu)$ is positive semi–definite for all $\mu' \in \hat{K}_{us}$.*

The standard way to compute if a real symmetric matrix M is positive semi–definite is: if and only if $\det(\mathcal{M}) \geq 0$ for all principal minors $\mathcal{M}$ of M. Note that this calculation needs to be done with rational numbers and can be somewhat expensive.

We only need to consider elements in the following region:

$$\mathcal{R} = \{\nu \in \mathfrak{a}_0^* \,|\, w_0\nu = -\nu \text{ and } (\lambda_M, \nu) \text{ is in the convex hull of } W\rho\} \tag{3}$$

The root hyperplanes $< (\lambda_M, \nu), \alpha\check{} > \in \mathbb{Z}$ break this region up into a finite number of facets, of various dimensions. The unitarity of $\overline{X}(\nu)$ is constant on each facet.

Proposition 4.3 *Fix $\mu \in \hat{K}_{us}$. The unitary representations of G with lowest K–type μ may be classified by the following algorithm.*

1. *Compute the facets of the region $\mathcal{R}$ (3). This is a finite set.*
2. *For each facet C choose a point $\nu_C \in C$ with $< \nu_C, \alpha\check{} > \in \mathbb{Q}$ for all roots α.*
3. *For each $\mu' \in \hat{K}_{us}$ compute the rational matrix $M_{\mu'}(\nu_C)$ as above.*
4. *The representations $\overline{X}(\nu)$ for $\nu \in C$ are if and only if $M_{\mu'}(\nu_C)$ is positive semi–definite for all $\mu' \in \hat{K}_{us}$.*

We will discuss an algorithm for computing the matrix $M_{\mu'}(\nu)$ in Section 6.

5 Improving the algorithm: using fewer K–types

The set $\hat{K}_{us}$ of unitarily small K–types is, in spite of its name, somewhat large.

Example 5.1 *If $G = Sp(2n, \mathbb{C})$ then $\mu = (a_1, \dots, a_n)$ is unitarily small if and only if μ is in the convex hull of the W–orbit of $(2n, 2n-2, \dots, 2)$.*

Basic Problem: Given μ, find a small set $S \subset \hat{K}_{us}$ such that: if π is an irreducible Hermitian representation with lowest K–type μ, then the Hermitian form is indefinite if and only if the signature is negative on the μ'–isotypic subspace of π for some $\mu' \in S$.

If S is such a set we will say S *detects unitarity* for μ. Proposition 3.2 says that $\hat{K}_{us}$ detects unitarity for all $\mu \in \hat{K}_{us}$.

Definition 5.2 *Suppose G is split over $\mathbb{R}$ or $\mathbb{C}$. Let $P = MAN$ be the minimal parabolic subgroup. Fix a K–type μ.*

If G is complex we say μ is petite *if $\mu^M \neq 0$ and 2α is not a weight of μ for any root α.*

Suppose G is real. If α is a root, consider the corresponding $SO(2)$, and the $SO(2)$ module generated by μ^M, the space of M–invariant vectors in μ. We say μ is petite *if $\mu^M \neq 0$ and these eigenvalues λ of this action all satisfy $|\lambda| \leq 2$.*

We write $\hat{K}_{petite}$ for the set of petite K–types.

The condition $\mu^M \neq 0$ is equivalent to the condition that μ is contained in the minimal principal series representations.

For example the trivial representation of K is petite, as is (every constitutent of) $\mathfrak{g}/\mathfrak{k}$ (this is irreducible if G is simple and G/K is not Hermitian symmetric).

Example 5.3 *Let $G = Sp(2n, \mathbb{C})$. A K–type is petite if and only if $\mu = (1, \dots, 1, 0, \dots, 0)$ or $(2, 1, \dots, 1, 0, \dots, 0)$, with an even number of ones in both cases.*

Example 5.4 *Let $G = GL(n,\mathbb{C})$. A K–type is petite if and only if $\mu = (a_1 - 1, \ldots, a_m - 1, -1, \ldots, -1)$ with $\sum a_i = n$, or is the dual of such a μ.*

We would like to say that the petite K–types detect unitarity for the trivial representation. This is not quite true.

Example 5.5 (Barbasch) *Let $G = Sp(4,\mathbb{C})$. Consider $X(t) = Ind_P^G(|det|^t \otimes 1)$ where $P = MN$ is the Siegel parabolic, so $M \simeq GL(2,\mathbb{C})$. Then $X(t)$ has infinitesimal character $(t + \frac{1}{2}, t - \frac{1}{2})$ in the usual coordinates. Let $\overline{X}(t)$ be the irreducible spherical constituent of $X(t)$. Then $\overline{X}(t)$ is the trivial representation for $t = \frac{3}{2}$, and is the spherical metaplectic representation for $t = 1$. It is unitary if and only if $|t| \leq 1$ or $t = \pm\frac{3}{2}$.*

Note that $X(t)$ is irreducible if $t = \frac{1}{2}$. This is surprising, since this is a potential reducibility point.

The K–types of $X(t)$ are $\{2a, 2b\}$ with $a \geq b \in \mathbb{Z}$. The petite K–types are the trivial representation and $\mu = (2,0)$. Since the spherical metaplectic representation (at $t = \frac{1}{2}$) is unitary and contains μ, we see that μ doesn't detect the non–unitarity of $\overline{X}(t)$ for $1 < t < \frac{3}{2}$. To detect the non–unitary of these representations we need to use $\mu' = (2,2)$.

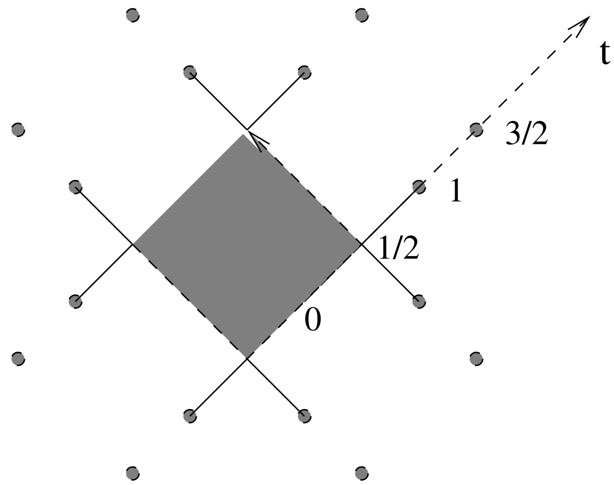

Diagram 1. Spherical unitary representations of $Sp(4,\mathbb{C})$. See also Diagram 3.

Proposition 5.6 (Barbasch) *For $Sp(2n,\mathbb{C})$ let $S \subset \hat{K}_{us}$ be the set of K–types of the form*

$$(2, \ldots, 2, 1, \ldots, 1, 0, \ldots, 0)$$

with an even number of ones. Then S detects unitarity for the trivial K–type.

Comparing this with Example 5.3 we see this is a good set of K–types: it detects unitarity for spherical representations, but is only slightly bigger than $\hat{K}_{petite}$.

Problem: Find a set, containing the petite K–types, but as small as possible, which detects unitarity for the trivial representation.

In any event, for petite K–types we have a Lefschetz principle, due to Barbasch and Salamanca.

Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ and let $X_{\mathbb{F}}(\nu)$ is a minimal principal series of a split group $G(\mathbb{F})$. Let w_0 be the long element of the Weyl group W. If $\nu \in \mathfrak{a}_0^*$ satisfies $w_0\nu = -\nu$ then $X_{\mathbb{F}}(\nu)$ has an invariant Hermitian form $(,)_{\mathbb{F}}(\nu)$. Let $\mu_{\mathbb{R}}$ be a petite K–type. Let τ be the corresponding representation of S on $\mu_{\mathbb{F}}^M$.

Now let $\mathbb{F}$ be a p–adic field, with ring of integers $\mathcal{O}$, and consider the split group $G(\mathbb{F})$. Then ν defines an unramified principal series representation $X_{\mathbb{F}}(\nu)$. This supports an invariant Hermitian form $(,)_{\mathbb{F}}(\nu)$ if $\nu \in \mathfrak{a}_0^*$ and $w_0\nu = -\nu$. Now τ defines a representation $\mu_{\mathbb{F}}$ of $K = G(\mathcal{O})$, containing the trivial representation of the Iwahori subgroup I.

Theorem 5.7 (Barbasch, Salamanca and Vogan) *The signatures of the Hermitian forms $(,)_{\mathbb{F}}(\nu)$ on $X_{\mathbb{F}}(\nu)(\mu_{\mathbb{F}})$ are all equal, for $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or a p–adic field.*

Recall by this we mean the associated Hermitian forms on $Hom_K(\mu_{\mathbb{F}}, X_{\mathbb{F}}(\nu))$ are equal (see Section 2). In particular these K–types have the same multiplicity in $\overline{X}_{\mathbb{F}}(\nu)$.

In section 7 we will compute $(,)_{\mathbb{F}}(\nu)$ in terms of the group algebra of the Weyl group.

5.1 Example: $Sp(2n, \mathbb{R})$

The definition of petite for real groups is somewhat cumbersome. Here we will compute the small set of K–types for $Sp(2n, \mathbb{R})$ satisyfing: the eigenvalues λ of the root $SO(2)$ subgroups acting on all of μ all satisfy $|\lambda| \leq 1$ (see Definition 5.2).

It appears that the set of K–types satisfying this condition are

$$(2, 2, \ldots, 0, 0, \ldots, 0)$$

$$(\overbrace{1, \ldots, 1}^{k}, 0, \ldots, 0, \overbrace{1, \ldots, 1}^{k})$$

6 Step Algebra

In the setting of Proposition 4.3 we are given $\mu \in \hat{K}_{us}$, a family of representations $X(\nu)$ with lowest K–type μ, and another K–type $\mu' \in \hat{K}_{us}$. We need to compute the Hermitian form on $X(\nu)(\mu')$. We will use the *step algebra* of Jouko Mikkelson.

We change notation and let $\mathfrak{g} \supset \mathfrak{k} \supset \mathfrak{b}_{\mathfrak{k}}$ be the complexified Lie algebras of G and K, and a Borel subalgebra of $\mathfrak{k}$ respectively. Write $\mathfrak{b}_{\mathfrak{k}} = \mathfrak{t} + \mathfrak{n}_{\mathfrak{k}}$ as usual.

Definition 6.1 *The step algebra:*

$$S = \{u \in \mathcal{U}(\mathfrak{g}) \,|\, \forall Y \in \mathfrak{n}_{\mathfrak{k}}, Yu \in \mathcal{U}(\mathfrak{g})\mathfrak{n}_{\mathfrak{k}}\}$$

Thus S is a subalgebra of $\mathcal{U}(\mathfrak{g})$ containing $\mathcal{U}(\mathfrak{g})^{\mathfrak{n}_\mathfrak{k}}$. It is immediate that if X is a $\mathcal{U}(\mathfrak{g})$ module then $X^{\mathfrak{n}_\mathfrak{k}}$ is stable under the action of S.

The main result is due to van den Hombergh [5]:

Theorem 6.2 *If X is an irreducible Harish–Chandra module then $X^{\mathfrak{n}_\mathfrak{k}}$ is an irreducible S–module.*

For the remainder of this section we fix an irreducible Harish–Chandra module X with lowest K–type μ.

Note that $Ad(\mathfrak{b}_\mathfrak{k})$ takes S to itself (check this). Therefore we can write

$$S = \oplus_{\mu\in\hat{T}} S_\mu$$

as a representation of T. Let $X^{\mathfrak{n}_\mathfrak{k}}(\gamma)$ be the γ weight space of T acting on X. Then

$$S_\mu : X^{\mathfrak{n}_\mathfrak{k}}(\gamma) \to X^{\mathfrak{n}_\mathfrak{k}}(\gamma+\mu)$$

The rest of this section is quite rough.

The idea is to find enough elements of S. Order the weights $\beta_1,\dots,\beta_r$ of T on $\mathfrak{p}$ so that if $\alpha>0$ is compact then $[X_\alpha, X_{\beta_i}] = \sum_{j<i} c^\alpha_{i,j} X_{\beta_j}$. Then β_1 is the highest weight in $\mathfrak{p}$, and $X_{\beta_1}\in S$. To see this let $\alpha>0$ be a compact root. Then

$$X_\alpha X_{\beta_1} = [X_\alpha, X_{\beta_1}] + X_{\beta_1} X_\alpha$$

and the term in brackets is 0 since $\alpha+\beta_1$ is not a root.

Now consider $\beta_2 = \beta_1 - \alpha$ for some compact simple root α. Then X_{β_1} is not in S since

$$\begin{aligned} X_\alpha X_{\beta_2} &= [X_\alpha, X_{\beta_2}] + X_{\beta_2} X_\alpha \\ &= cX_{\beta_1} + X_{\beta_2} X_\alpha (c\neq 0) \end{aligned}$$

We may assume $[X_\alpha, X_{\beta_2}] = X_{\beta_1}$. We also assume $X_\alpha, X_{-\alpha}, H_\alpha$ is a standard triple.

The error term has T–weight $\beta_1-\alpha=\beta_2$. So let

$$Y_{\beta_2} = X_{\beta_2} + cX_{-\alpha} X_{\beta_1}$$

for some constant c to be determined.

We compute

$$\begin{aligned} X_\alpha Y_{\beta_2} &= X_\alpha X_{\beta_2} + cX_\alpha X_{-\alpha} X_{\beta_1} \\ &= (X_{\beta_2} X_\alpha + X_{\beta_1}) + cX_\alpha X_{-\alpha} X_{\beta_1} \\ &= X_{\beta_2} X_\alpha + X_{\beta_1} + cH_\alpha X_{\beta_1} + cX_{-\alpha} X_\alpha X_{\beta_1} \\ &= X_{\beta_2} X_\alpha + X_{\beta_1} + cH_\alpha X_{\beta_1} + cX_{-\alpha} X_{\beta_1} X_\alpha \end{aligned}$$

We can ignore the first and last terms since they have X_α on the right. We conclude that

$$X_\alpha Y_{\beta_2} \in X_{\beta_1} + cH_\alpha X_{\beta_1} + \mathcal{U}(\mathfrak{g})\mathfrak{n}_\mathfrak{k}$$

So formally we would like to take "$c = -\frac{1}{H_\alpha}$". We might try the element $H_\alpha X_{\beta_2} - X_{-\alpha}X_{\beta_1}$ which is obtained by formally taking $c = -\frac{1}{H_\alpha}$ and multiplying by H_α. However there is another term of weight β_2 necessary. Let

$$Y_{\beta_2} = H_\alpha X_{\beta_2} - X_{-\alpha}X_{\beta_1} - 2X_{\beta_2}.$$

Then a similar calculation to the preceding one shows that

$$Y_{\beta_2} \in S$$

Continuing in this way for $i \leq r$ we obtain an element $Y_{\beta_i} \in S$ of the form

$$Y_{\beta_i} = AX_{\beta_i} + \sum_{j<i} B_j X_{\beta_j}$$

with $A, B_j \in \mathcal{U}(\mathfrak{k})$.

We expect that the algebra generated by the Y_{β_i} acts irreducibly on $X^{\mathfrak{n}_\mathfrak{k}}$, even though it doesn't generate S. For $\vec{i} = (i_1, \ldots, i_n)$ let $Y_{\vec{i}} = Y_{\beta_{i_1}} \ldots Y_{\beta_{i_n}}$. Let $\beta(\vec{i}) = \sum_j \beta_{i_j}$. Then

$$Y_{\vec{i}} X(\mu) \to X(\mu + \beta(\vec{i}))$$

Let v_μ be a highest weight vector $X(\mu)$, and fix another K–type μ'. Then we expect that

$$\{Y_{\vec{i}} v_\mu \mid \beta(\vec{i}) = \mu' - \mu\}$$

will span the space of highest weight vectors of $X(\mu')$.

6.1 Computing the Hermitian Form

Now assume X has an invariant Hermitian form $<,>$. Fix a K–type μ' of X. We use the step algebra to compute $<,>$ on the μ'–isotypic subspace.

Let $*$ be the anti–automorphism of $\mathfrak{g}$, given by $X^* = -\overline{X}$ for $X \in \mathfrak{g}(\mathbb{C})$. Then the invariant Hermitian form satisfies $< Xv, w > = < v, X^* w >$ for all $X \in \mathcal{U}(\mathfrak{g}), v, w \in X$.

Now assume $v = v_\mu$ is the highest weight vector of the lowest K–type. Choose $\vec{i}, \vec{i}'$ with $\beta(\vec{i}) = \beta(\vec{i}') = \mu' - \mu$, and let $w = Y_{\vec{i}} v, w' = Y_{\vec{i}'} v$. To compute $<,>$ on $X(\mu')$ it is enough to compute $< w, w' >$ for all such w, w'. We have

$$< w, w' > = < Y_{\vec{i}} v, Y_{\vec{i}'} v > = < Y_{\vec{i}'}^* Y_{\vec{i}} v, v >$$

Now $Y_{\vec{i}'}^* Y_{\vec{i}} \in S(0)$. Note that $S(0)$ contains $\mathcal{U}(\mathfrak{g})^K$. We are reduced to the following computation:

Problem: Suppose we are given a Harish–Chandra module X with lowest K–type μ. Assume μ has multiplicity one, for example assume X is a standard or irreducible module. Then $X(\mu)^{\mathfrak{n}_\mathfrak{k}}$ is one–dimensional, and $S(0)$ acts by an algebra homomorphism on it. Compute this algebra homomorphism.

We proceed in two steps. First we assume $X = R_{\mathfrak{q}}(X_L)$ is a standard module, constructed by cohomological induction from $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$, with L quasisplit. By a generalization of [7] (which treats the case of $\mathcal{U}(\mathfrak{g})^K$) there is a homomorphism $S(0) \to S_{\mathfrak{l}}(0)$, where $S_{\mathfrak{l}}$ is the corresponding algebra for $\mathfrak{l}$. The lowest K–type of X_L is $\mu_L = \mu - 2\rho(\mathfrak{u} \cap \mathfrak{p})$, and we are reduced to calculating the action of $S_{\mathfrak{l}}(0)$ on $X_L(\mu_L)$.

This reduces us to the case of G quasisplit, and $P = MAN$ the minimal parabolic subgroup of G.

We first consider $\mathcal{U}(\mathfrak{g})^K$. Write

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}) \otimes S(\mathfrak{a}) \otimes \mathcal{U}(\mathfrak{K}).$$

Consider the map $\phi : \mathcal{U}(\mathfrak{g}) \to S(\mathfrak{a}) \otimes \mathcal{U}(\mathfrak{k})$ which takes $\mathfrak{n}\mathcal{U}(\mathfrak{g})$ to 0.

The next two lemmas are due to Lepowsky [4].

Lemma 6.3 *$\phi : \mathcal{U}\mathfrak{g})^K \to [S(\mathfrak{a}) \otimes \mathcal{U}(\mathfrak{k})]^M$ is an algebra anti–automorphism.*

Now fix a principal series representation $X = Ind_P^G(\delta \otimes \nu \otimes 1)$ where $P = MAN$, $\delta \in \hat{M}$ and $\nu \in \mathfrak{a}^*$. Suppose X has lowest K–type μ, so $\mu|_M$ contains δ with multiplicity 1. We obtain a homomorphism $\lambda_1 : \mathcal{U}(\mathfrak{k})^M \to \mathbb{C}$ be the action of $\mathcal{U}(\mathfrak{k})^M$ on the δ–isotypic subspace of $\mu|_M$. We also have a homomorphsim $\lambda_2 : S(\mathfrak{a}) \to \mathbb{C}$ given by evaluation at $\nu + \rho$ (check the shift).

Lemma 6.4 ([7]) *The action of $\mathcal{U}(\mathfrak{g})^K$ on $X(\mu)^{\mathfrak{n}\mathfrak{k}}$ is by $(\lambda_1 \otimes \lambda_2) \circ \phi$.*

We expect a similar result for $S(0)$.

7 Spherical Representations and the Weyl group

In this section we give an alternative method for classifying unitary representations with a given lowest K–type in the case that the representations are spherical. We consider p-adic fields also.

The main point is to reduce the calculation to one in the group algebra of the Weyl group. We start with a formal construction for the Weyl group.

Let V be a real vector space with a positive definite symmetric inner form $(,)$, a root system R and R^+ a choice of positive roots. Let W be the Weyl group of R and S the simple reflections in W corresponding to R^+. Let w_0 be the long element of the Weyl group. Then $w_0^2 = 1$ and $w_0 = s_N s_{n-1} \dots s_1$ where $N = |R^+|$ and $s_i \in S$ for all i.

Suppose $\nu \in V$ is dominant, i.e. $< \nu, \alpha^\vee > \geq 0$ for all $\alpha \in R$ where $<,>$ is the natural pairing between V and V^*. Recall $\alpha^\vee$ is the corresponding element of $R^\vee \subset V^*$; equivalently $< \nu, \alpha^\vee >= 2(\nu, \alpha)/(\alpha, \alpha)$. We associate to ν an element $A(\nu)$ in the real group algebra $\mathbb{R}[W]$ of W.

Definition 7.1 *(1) For $s \in S$ and $\nu \in V$, define*

$$A_s(\nu) = \frac{1}{1+ < \nu, \alpha^\vee >} e + \frac{< \nu, \alpha^\vee >}{1+ < \nu, \alpha^\vee >} s$$

where e is the identity in W. (2) For $\nu \in V$ define

$$A(\nu) = A_{s_N}(s_{N-1}\dots s_1\nu)\dots A_{s_2}(s_1\nu)A_{s_1}(\nu) \in R[W]$$

If (τ, E_τ) is any representation W, we obtain $\tau(A_s(\nu)) \in End(E_\tau)$, and

$$\tau(A_s(\nu))v = \begin{cases} v & \tau(s)v = v \\ \frac{1-<\nu,\alpha\check{}>}{1+<\nu,\alpha\check{}>}v & \tau(s)v = -v \end{cases}$$

Furthermore $A_s(0) = 1$ and $A_s(\nu) \to s$ as $<\nu, \alpha\check{}> \to +\infty$. Finally note that $A_{s_\alpha}(\nu)$ is invertible unless $<\nu, \alpha\check{}>= 1$ for some $\alpha \in R^+$.

Lemma 7.2 *1. $A(\nu)$ is a rational function of ν*

2. $A(0) = I$

3. $\lim_{\nu\to\infty} A(\nu) = w_0$

4. $A(\nu)$ is invertible in $\mathbb{R}[W]$ provided $<\nu,\alpha\check{}>\neq 1$ for all α.

Theorem 7.3 *Consider the element $A(\nu) \in \mathbb{R}[W]$, for $\nu \in V$.*

1. *$A(\nu)$ is well defined (independent of the choice of reduced expression for w_0) provided $P(\nu) \neq 0$. In particular this holds for all (weakly) dominant ν, i.e. satisfying $<\nu,\alpha\check{}>\geq 0$ for all positive roots α.*
2. *$A(\nu)$ is invertible if $<\nu,\alpha\check{}>\neq 1$ for all $\alpha \in R^+$.*
3. *$A(\nu)$ is self–adjoint if $w_0\nu = -\nu$.*

Example:

Let R be of type C_2 and let τ be the reflection representation. Write $\nu = (x, y)$. Then

$$A_\tau(\nu) = P(\nu)^{-1}\begin{pmatrix} 1 & x-y \\ x-y & 1\end{pmatrix}\begin{pmatrix} 1+x & 0 \\ 0 & 1-y\end{pmatrix}\begin{pmatrix} 1 & x+y \\ x+y & 1\end{pmatrix}\begin{pmatrix} 1+y & 0 \\ 0 & 1-y\end{pmatrix}$$

$$= P(\nu)^{-1}\begin{pmatrix} (1+y)(1+x)+ (1+y)(1-x)(x-y)(x+y) & 2x(y^2-1) \\ 2x(y^2-1) & (1-y)(1-x)+ (1-y)(1+x)(x-y)(x+y)\end{pmatrix}$$

where

$$P(\nu) = (1+x)(1+x+y)(1+y)(1+x-y).$$

It is interesting to note that the eigenvalues of $A_\tau(\nu)$ are not rational functions of ν. In this example the eigenvalues of $P(\nu)A_\tau(\nu)$ are

$$[1 + x^2 + xy - x^3y - y^2 + xy^3]\pm$$
$$[5x^2 - 2x^4 + x^6 + 2xy - 2x^5y + y^2 - 4x^2y^2-$$
$$x^4y^2 + 4x^3y^3 - 2y^4 + 3x^2y^4 - 2xy^5 + y^6]^{\frac{1}{2}}$$

Question: For each $\tau \in \hat{W}$ and each dominant ν satisfying $w_0\nu = -\nu$, determine if $A_\tau(\nu)$ is positive semi–definite.

It is clear that whether $a_\tau(\nu)$ is positive semi–definite can only change at values of ν for which $< \nu, \alpha\check{} >= 1$ for some $\alpha \in R^+$. Divide the dominant chamber up into a finite number of regions, cut out by the N hyperplanes given by $< \alpha, \nu >= 1$ for $\alpha \in R^+$. This gives a finite number of regions to check.

Example: $Sp(4)$ The hyperplanes $< \nu, \alpha\check{} >= 1$ cuts the dominant chamber for $Sp(4)$ into 7 regions. Two of these are positive semi–definite for the reflection representation of W. These are shaded gray in Diagram 1.

For one of the two–dimensional representations of W the second region in Diagram 1 is not positive semi–definite. This leaves the closue of the first region, and the isolated point $(2, 1) = \rho$ as the region of unitarity for the spherical representation over $\mathbb{R}, \mathbb{C}$ or a p–adic field. This is shown in Diagram 2.

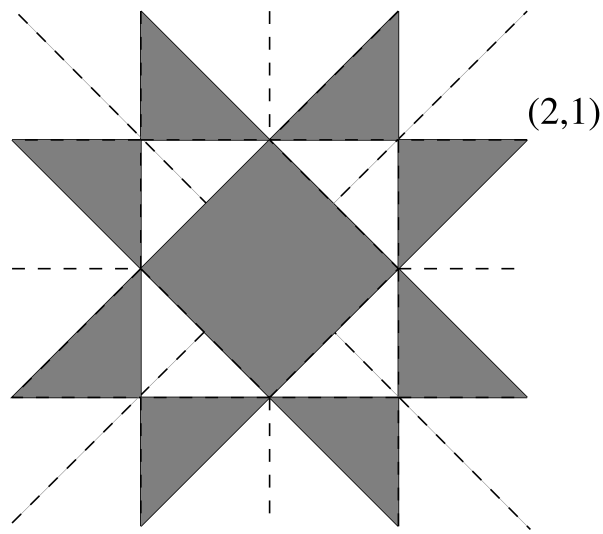

Diagram 2. Positive semi–definite regions for the reflection representation of W for $Sp(4)$. Root lines are shown as (dotted) lines.

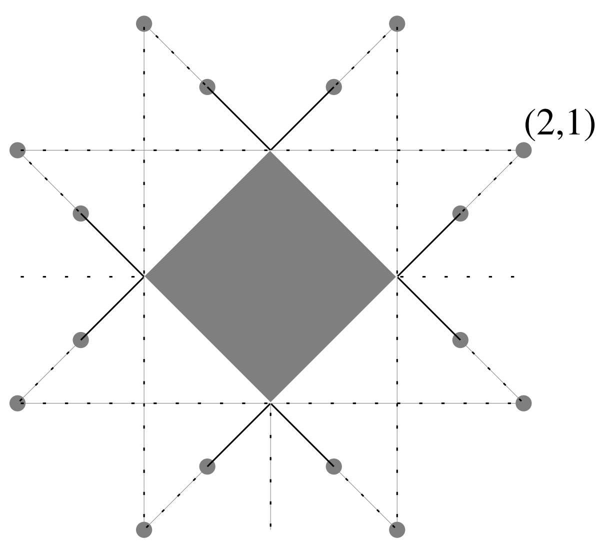

Diagram 3. Unitary regions for the spherical representation of of $Sp(4,\mathbb{C})$. The pictures for $\mathbb{R}$ and a p–adic field are is the same with the line segments removed. Lines where a coroot equals $0, \pm 1$ are shown as dotted lines, solid lines indicate unitarity.

Now let G be a split group over the p–adic field $\mathbb{F}$ as at the end of Section 5. Choose a minimal parabolic subgroup $P = MN$ with $M \simeq \mathbb{F}^{*n}$. Let V be the space of positive real valued characters of M. This contains the root system of G, where α corresponds to the character $g \to |\alpha(g)|$ $(g \in M)$. Associated to $\nu \in V$ is the unramified principal series representation $X(\nu)$, with unique irreducible quotient $\overline{X}(\nu)$. Let w_0 be the longest element of the Weyl group.

Theorem 7.4 (Barbasch Moy [2]) *The irreducible representation $\overline{X}(\nu)$ is unitary if and only if*

1. $w_0(\nu) = -\nu$
2. $A_\tau(\nu)$ *is positive semi–definite for all* $\tau \in \widehat{W}$

Together with Theorem 5.7 we obtain a unitarity criterion for spherical representations of real or complex groups. Let $P = MAN$ be a minimal parabolic subgroup. Choose $\nu \in \mathfrak{a}_0^*$ with $w_0\nu = -\nu$, and let $\overline{X}(\nu)$ be the corresponding irreducible spherical representation.

Theorem 7.5 (Barbasch/Salamanca/Vogan) *Let μ be a petite K–type, and let τ be the corresponding representation of W on the 0–weight space of μ in the complex case, or μ^M in the real case. If $\overline{X}(\nu)$ is unitary then $A_\tau(\nu)$ is positive semi–definite.*

Note that this is actually a non–unitary test: it can be used to prove representations are not unitary. As discussed in Example 5.5 this is not enough to classify the unitary representations. For $\overline{X}(\nu)$ to be unitary it is necessary, but not sufficient, for $A_\tau(\nu)$ to be positive semi–definite for all τ arising as above for μ petite.

The K–type $\mu' = \mathfrak{g}/\mathfrak{p}$ is petite, and gives the reflection representation of W. The rank of $A_\tau(\nu)$ gives the multiplicity of μ' in $\overline{X}(\nu)$, and has been computed by Joseph and Bang–Jensen. Perhaps computing the signature isn't much harder.

In practice the reflection representation alone shows that many representations are not unitary.

References

[1] D. Vogan A. Knapp. *Cohomological Induction and Unitary Representations.* Princeton University Press, Princeton, 1995.

[2] Dan Barbasch and Allen Moy. A unitarity criterion for p-adic groups. *Invent. Math.*, 98(1):19–37, 1989.

[3] A. Knapp. *Representation Theory of Semisimple Groups. An overview based on.* Princeton University Press, Princeton, NJ, 1986.

[4] J. Lepowsky. Algebraic results on representations of semisimple Lie groups. *Trans. Amer. Math. Soc.*, 176:1–44, 1973.

[5] A. van den Hombergh. A note on Mickelsson's step algebra. *Nederl. Akad. Wetensch. Proc. Ser. A* **78**=*Indag. Math.*, 37:42–47, 1975.

[6] D. Vogan. *Representations of Real Reductive Lie Groups*, volume 15 of *Progress in mathematics.* Birkhäuser, Boston, 1981.

[7] David A. Vogan, Jr. The algebraic structure of the representation of semisimple Lie groups. I. *Ann. of Math. (2)*, 109(1):1–60, 1979.