Computer Computations in Representation Theory III: Unitary Representations of real Lie groups

With an appendix on mathematical software

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6 Recap

In Lecture I we showed that $SL(2,\mathbb{R})$ has interesting infinite dimensional unitary representations. That is we found representations π_ν on a Hilbert space $L^2(\mathbb{R})$ for which the operators π_ν are unitary, i.e. $(\pi_\nu(g)v, \phi_\nu(g)w) = (v, w)$ for all $v, w \in L^2(\mathbb{R})$ and $g \in SL(2,\mathbb{R})$. Here $\nu \in i\mathbb{R}$ (with $(,)$ the usual inner product) or $-1 \leq \nu \leq 1$ (with a different inner product).

We are interested in similar representations of other Lie groups such as $GL(n,\mathbb{R}), Sp(2n,\mathbb{R})$

and $SO(n,\mathbb{R})$. We are also interested in the exceptional groups E_6, E_7, E_8, F_4 and G_2.

Each of the groups under discussion contains a subgroup T isomorphic to $\mathbb{R}^{*n}$, which gives rise to the root system $R \subset \mathbb{R}^n$ and the Weyl group W. Recall R is a finite set of roots satisfying certain conditions. In particular if $\alpha, \beta \in R$ then

$$< \alpha, \beta >= \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$$

and the corresponding reflection

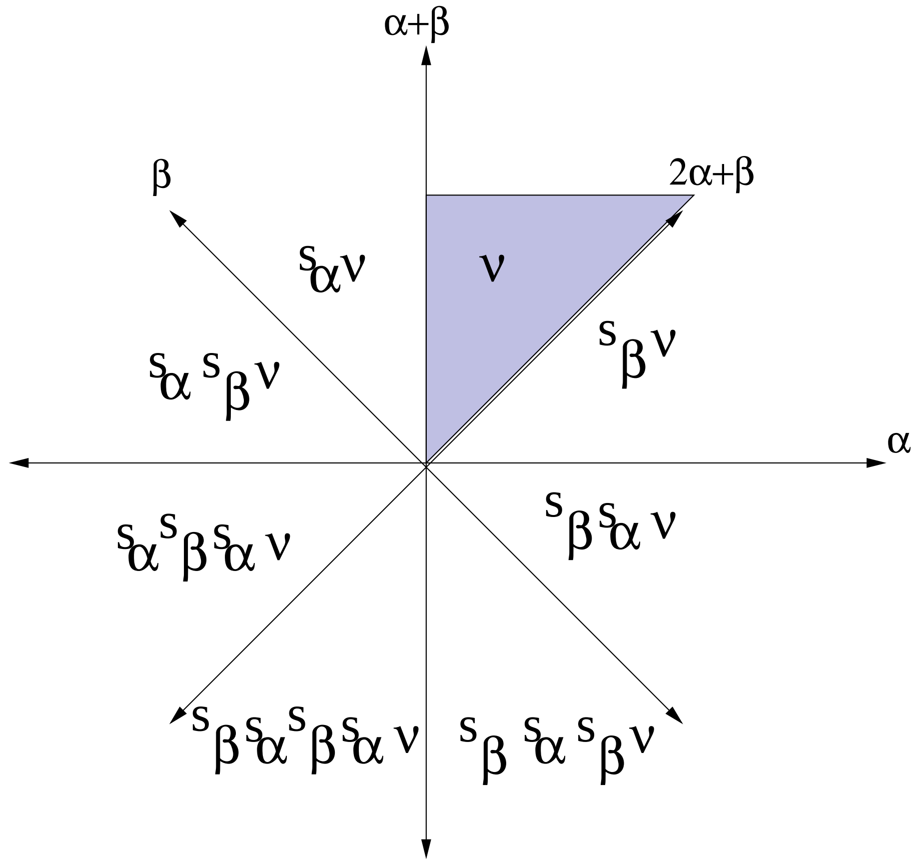
$$\sigma_\alpha(v) = v- < v, \alpha > \alpha$$

takes R to itself. The Weyl group is the group generated by the reflections σ_α $(\alpha \in R)$. This acts on $\mathbb{R}^n$.

Recall a set of simple roots $\Delta \subset R$ is a basis of V satisfying certain other condition.

We say ν is *dominant* if $< \nu, \alpha > \geq 0$ for all $\alpha \in \Delta$.

Lemma 6.1 *Given ν there exists $w \in W$ such that $w\nu$ is dominant.*



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7 Representations

Let G be one of our groups. The diagonal subgroup T is isomorphic to $\mathbb{R}^* \times \cdots \times \mathbb{R}^*$. I'll assume the number of factors is n, i.e. $T \simeq \mathbb{R}^{*n}$. For example take G to be $GL(n, \mathbb{R})$, $Sp(2n, \mathbb{R})$, $SO(2n, \mathbb{R})$ or $SO(2n+1, \mathbb{R})$.

Note that if $\nu = (z_1, \ldots, z_n) \in \mathbb{C}^n$ then ν gives a one–dimensional representation ν of $T \simeq \mathbb{R}^{*n}$ by

$$\nu(x_1, \ldots, x_n) = |x_1|^{z_1} \ldots |x_n|^{z_n}$$

Lemma 7.1 *For every $\nu \in \mathbb{C}^n$ there is an* *irreducible representation, denoted* π_ν, *of* G. *It is unitary if (but not only if!)* $\nu \in i\mathbb{R}^n$. *Finally,* $\pi_\nu \simeq \pi_{w\nu}$ *for all* $w \in W$.

By the last statement we may as well assume ν is dominant.

The representation π_ν is on a space of functions $\mathcal{F}$ on G. It is an example of an *induced representation.* In the case of $SL(2, \mathbb{R})$ these are the representations of Lecture I.

Recall in the case of $SL(2, \mathbb{R})$ π_ν is unitary if $\nu \in i\mathbb{R}$ or $-1 \leq \nu \leq 1$. The interesting and hard case is the latter one.

From now on we assume ν is *real*, i.e. $\nu \in \mathbb{R}^n$.

In our setting we are interested in the following problem:

Problem: For which $\nu \in \mathbb{R}^{*n}$ is π_ν unitary?

Recall the Weyl group W acts on $\mathbb{R}^n$, and w_0 is the longest element of W.

Lemma 7.2 *If π_ν is unitary then $w_0\nu = -\nu$.*

In some cases $w_0 = -1$, in which case this is no condition.

8 A calculation in W

We now give a formal construction in the Weyl group which will answer the question posed in the previous section.

Let $V = \mathbb{R}^n$, with the standard inner product $(v, w) = v \cdot w$. Suppose $R \subset \mathbb{R}^n$ is a root system, and W is the Weyl group of R. Pick a set Δ of simple roots. Write

$$w_0 = s_N s_{N-1} \dots s_1$$

where $s_i = s_\alpha$ for some $\alpha \in \Delta$. Here $N = \frac{1}{2}|N|$.

Fix a representation τ of W of dimension m.

Definition 8.1 *Suppose $\nu \in V$.*

(1) For $\alpha \in \Delta$ define

$$A_{\pi,\alpha}(\nu) = \pi(1) + <\nu, \alpha> \pi(s_\alpha)$$

(2) Define

$$A_\pi(\nu) = A_{\alpha_N}(s_{\alpha_{N-1}} \dots s_{\alpha_1}\nu) \dots A_{\alpha_2}(s_{\alpha_1}\nu) A_{\alpha_1}(\nu)$$

Thus $A(\nu)$ is an $m \times m$ matrix. Note that $\pi(s_\alpha) = diag(\pm 1, \dots, \pm 1)$ in some basis, so

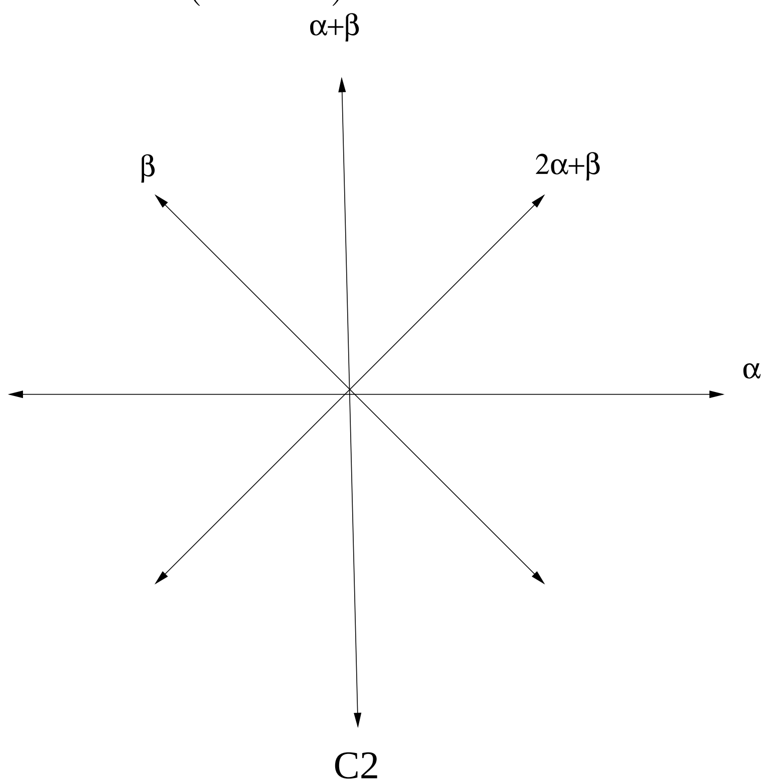
$$A_{\pi,\alpha} = \begin{pmatrix} 1 \pm <\nu, \alpha> & & \\ & \ddots & \\ & & 1 \pm <\nu, \alpha> \end{pmatrix}$$

In particular $A_{\pi,\alpha}$ is invertible if $<\nu, \alpha> \neq \pm 1$.

Example: $Sp(4, \mathbb{R})$

We'll compute $A_\pi(\nu)$ where π is the reflection representation.

This group has type C_2, and the root system looks like this ($n = 2$):



The simple roots are α, β. Write $\nu = (x, y)$.

We have s_α is reflection in the x coordinate, and

s_β is reflection in the line $x = y$:

$$s_\alpha(x, y) = (-x, y) \quad s_\beta(x, y) = (y, x)$$

That is, the reflection representation π is given

by matrices

$$\pi(s_\alpha) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\pi(s_\beta) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Also note that

$$< \nu, \alpha >= 2(2x)/4 = x$$

$$< \nu, \beta >= 2(x-y)/2 = x - y$$

As noted in Lecture II, $w_0 = s_\beta s_\alpha s_\beta s_\alpha$. Therefore $A_\pi(\nu)$ is the following product:

$$A_{\pi,s_\beta}(s_\alpha s_\beta s_\alpha \nu) A_{\pi,s_\alpha}(s_\beta s_\alpha \nu) A_{\pi,s_\beta}(s_\alpha \nu) A_{\pi,s_\alpha}(\nu)$$

Thus

$$A_{\pi,s_\alpha}(\nu) = \pi(1) + < \nu, \alpha > \pi(s_\alpha)$$

$$= \pi(1) + x\pi(s_\alpha)$$

$$= \begin{pmatrix} 1-x & 0 \\ 0 & 1+x \end{pmatrix}$$

Next $< s_\alpha \nu, \beta >=< (-x, y), (1, -1) > -x-$

y, and

$$A_{\pi,s_\beta}(s_\alpha\nu) = \pi(1) + < s_\alpha\nu, \beta > \pi(s_\beta)$$

$$= \pi(1) + (-x-y)\pi(s_\beta)$$

$$= \begin{pmatrix} 1 & -x-y \\ -x-y & 1 \end{pmatrix}$$

For the next step

$$< s_\beta s_\alpha\nu, \alpha > = < (y,-x),(2,0) > = y$$

and

$$A_{\pi,s_\alpha}(s_\beta s_\alpha\nu) = \pi(1) + < s_\beta s_\alpha\nu, \alpha > \pi(s_\alpha)$$

$$= \pi(1) + y\pi(s_\alpha)$$

$$= \begin{pmatrix} 1-y & 0 \\ 0 & 1+y \end{pmatrix}$$

Finally $< s_\alpha s_\beta s_\alpha\nu, \beta > = < (-y,-x),(1,-1) >$

$y - x$, and

$$A_{\pi,s_\beta}(s_\alpha \nu) = \pi(1) + < s_\alpha s_\beta s_\alpha \nu, \beta > \pi(s_\beta)$$

$$= \pi(1) + (-x - y)\pi(s_\beta)$$

$$= \begin{pmatrix} 1 & y - x \\ y - x & 1 \end{pmatrix}$$

This gives

$$A_\pi(\nu) = \begin{pmatrix} 1 & y - x \\ y - x & 1 \end{pmatrix} \begin{pmatrix} 1 - y & 0 \\ 0 & 1 + y \end{pmatrix}$$
$$\begin{pmatrix} 1 & -x - y \\ -x - y & 1 \end{pmatrix} \begin{pmatrix} 1 - x & 0 \\ 0 & 1 + x \end{pmatrix}$$

or

$$\begin{pmatrix} \begin{matrix} (1+y)(1+x)+ \\ (1+y)(1-x)(x-y)(x+y) \end{matrix} & 2x(y^2-1) \\ 2x(y^2-1) & \begin{matrix} (1-y)(1-x)+ \\ (1-y)(1+x)(x-y)(x+y) \end{matrix} \end{pmatrix}$$

Definition 8.2 *A real symmetric matrix is positive semi–definite if its eigenvalues are all* ≥ 0*.*

Remark: The eigenvalues of a real symmetric matrix are real.

Example:

$$X := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 7 \end{pmatrix}$$

According to Mathematica the eigenvalues are:

$$\frac{11}{3} + \frac{23\,5^{\frac{2}{3}}}{3\left(241 + 9\,i\,\sqrt{34}\right)^{\frac{1}{3}}} + \frac{\left(5\,\left(241 + 9\,i\,\sqrt{34}\right)\right)^{\frac{1}{3}}}{3}$$

$$\frac{11}{3} - \frac{23\,5^{\frac{2}{3}}\,\left(1 + i\,\sqrt{3}\right)}{6\left(241 + 9\,i\,\sqrt{34}\right)^{\frac{1}{3}}} - \frac{\left(1 - i\,\sqrt{3}\right)\,\left(5\,\left(241 + 9\,i\,\sqrt{34}\right)\right)^{\frac{1}{3}}}{6}$$

$$\frac{11}{3} - \frac{23\,5^{\frac{2}{3}}\,\left(1 - i\,\sqrt{3}\right)}{6\left(241 + 9\,i\,\sqrt{34}\right)^{\frac{1}{3}}} - \frac{\left(1 + i\,\sqrt{3}\right)\,\left(5\,\left(241 + 9\,i\,\sqrt{34}\right)\right)^{\frac{1}{3}}}{6}$$

Numerically:

$$\begin{array}{r} 10.79 + 0.i \\ -0.34 + 4.44 \times 10^{-16} i \\ 0.54 - 4.44 \times 10^{-16} i \end{array}$$

Moral: Don't trust what computers tell you.

Theorem 8.3 *1. $A_\pi(\nu)$ is well–defined*

2. *$A_\pi(\nu)$ is a polynomial function in ν*
3. *$A_\pi(\nu)$ is invertible unless $<\nu,\alpha>=\pm 1$ for some root α*
4. *$A_\pi(\nu)$ is symmetric if $w_0\nu=-\nu$*

Conjecture 8.4 *Assume $\nu\in\mathbb{R}^n$ and $w_0\nu=-\nu$. Then π_ν is unitary if and only if $A_\pi(\nu)$ is positive semi–definite for all irreducible representations π of W.*

This is almost certainly true. It is true for classical groups. If it isn't true it would be good to know why.

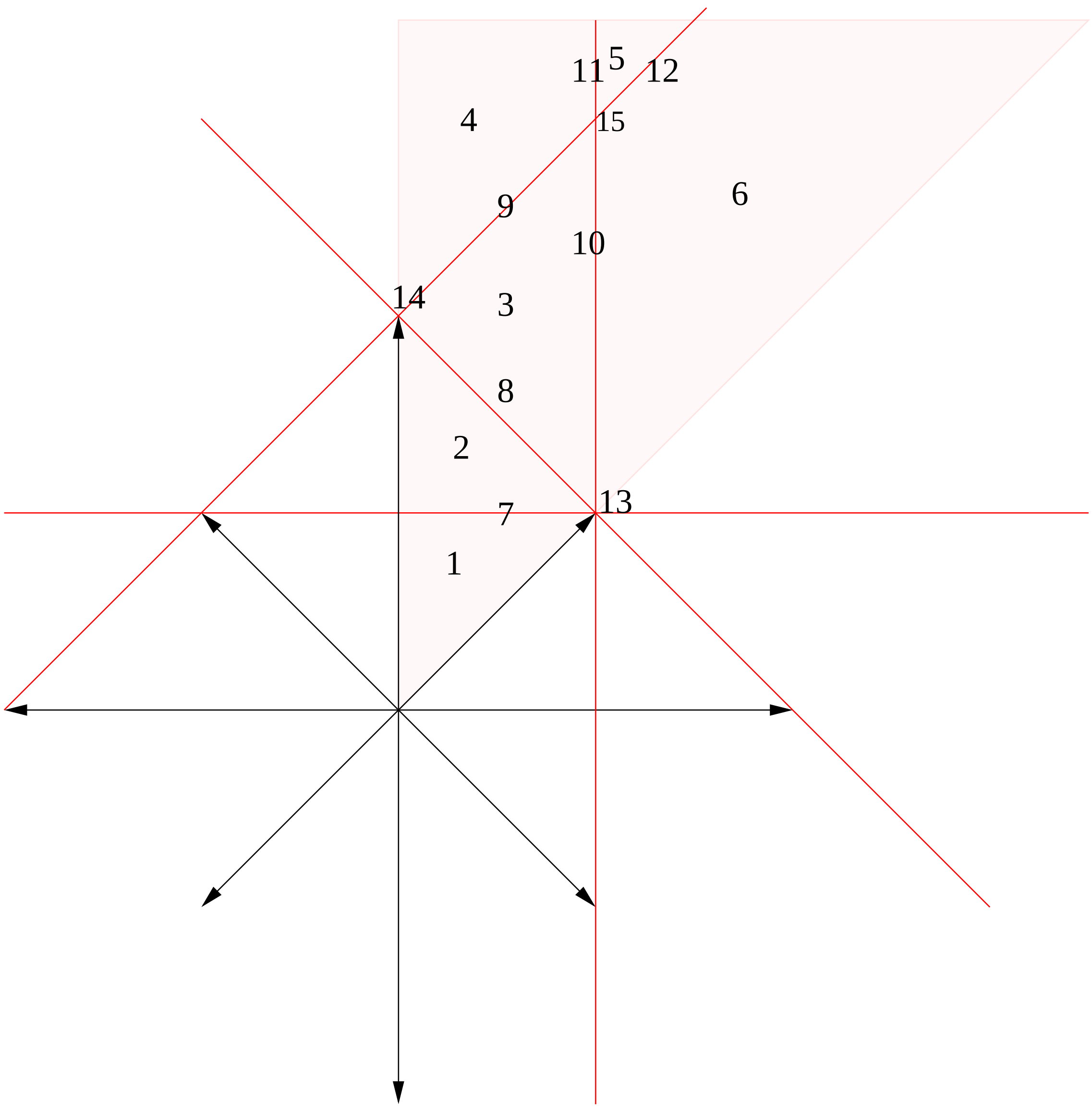
Theorem 8.5 *The conjecture is true if you replace $\mathbb{R}$ with a "p–adic" field.*

Problem: Compute the set of ν for which $A_\pi(\nu)$ is positive definite for all π. Given the conjecture this answers the Problem stated earlier.

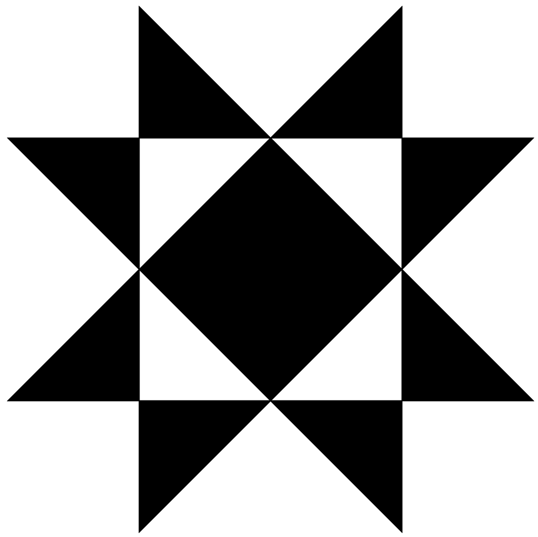
We may assume ν is dominant. The eignevalues of $A_\pi(\nu)$ are continuous functions of ν; they can only change sign at where $<w\nu,\alpha>=1$ for some $\alpha\in\Delta$ and $w\in W$.

We therefore divide the dominant region into a finite number of facets: those cut out by the planes $< v, \alpha >= 1$ $(\alpha \in \Delta)$.

Example: $Sp(4, \mathbb{R})$



Here are the regions for which $A_\pi(\nu)$ is positive semidefinite where π is the reflection representation:

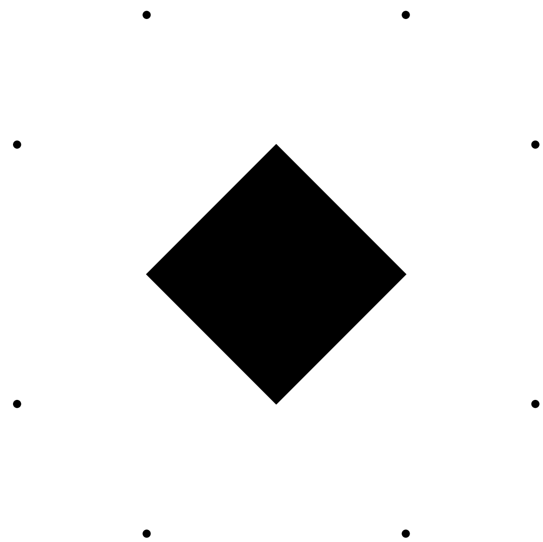


The eigenvalues of $A_\pi(\nu)$ are:

$$[1 + x^2 + xy - x^3y - y^2 + xy^3] \pm$$
$$[5x^2 - 2x^4 + x^6 + 2xy - 2x^5y + y^2 - 4x^2y^2 -$$
$$x^4y^2 + 4x^3y^3 - 2y^4 + 3x^2y^4 - 2xy^5 + y^6]^{\frac{1}{2}}$$

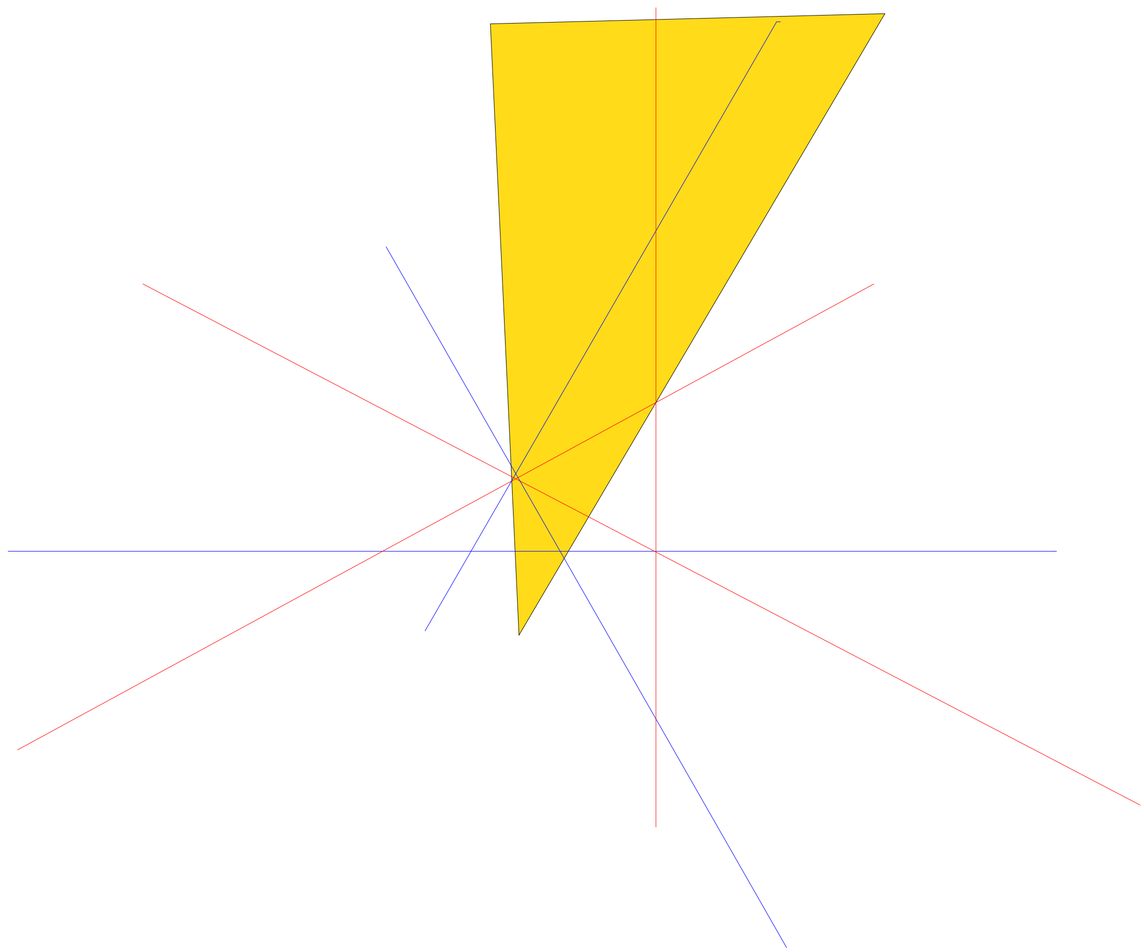
In the picture the (dominant) points where both eigenvalues are ≥ 0 are colored black (and made Weyl group invariant).

In all representations of W:



This is the picture of the unitary representations of $Sp(4,\mathbb{R})$.

Example: G_2



The facets for G_2

It suffices to use a reducible four-dimensional representation. The first eigvenvalue is:

$(54 - 108x^2 + 54x^4 + 144\sqrt{3}xy + 72\sqrt{3}x^5y + 36y^2 -$
$36x^2y^2 - 192\sqrt{3}xy^3 - 240\sqrt{3}x^3y^3 - 90y^4 + 72\sqrt{3}xy^5 -$
$[(54 - 108x^2 + 54x^4 + 144\sqrt{3}xy + 72\sqrt{3}x^5y + 36y^2 - 36x^2y^2 -$
$192\sqrt{3}xy^3 - 240\sqrt{3}x^3y^3 - 90y^4 + 72\sqrt{3}xy^5)^2 + 108(-27 + 216x^2 -$
$594x^4 + 756x^6 - 459x^8 +$
$108x^10 + 216y^2 - 1188x^2y^2 + 1332x^4y^2 - 108x^6y^2 - 108x^8y^2 -$
$144x^10y^2 - 594y^4 + 2892x^2y^4 - 2178x^4y^4 + 216x^6y^4 + 960x^8y^4 +$
$652y^6 - 2796x^2y^6 + 1224x^4y^6 - 1888x^6y^6 - 267y^8 + 828x^2y^8 +$
$960x^4y^8 + 36y^10 - 144x^2y^10)]^{\frac{1}{2}})/54$

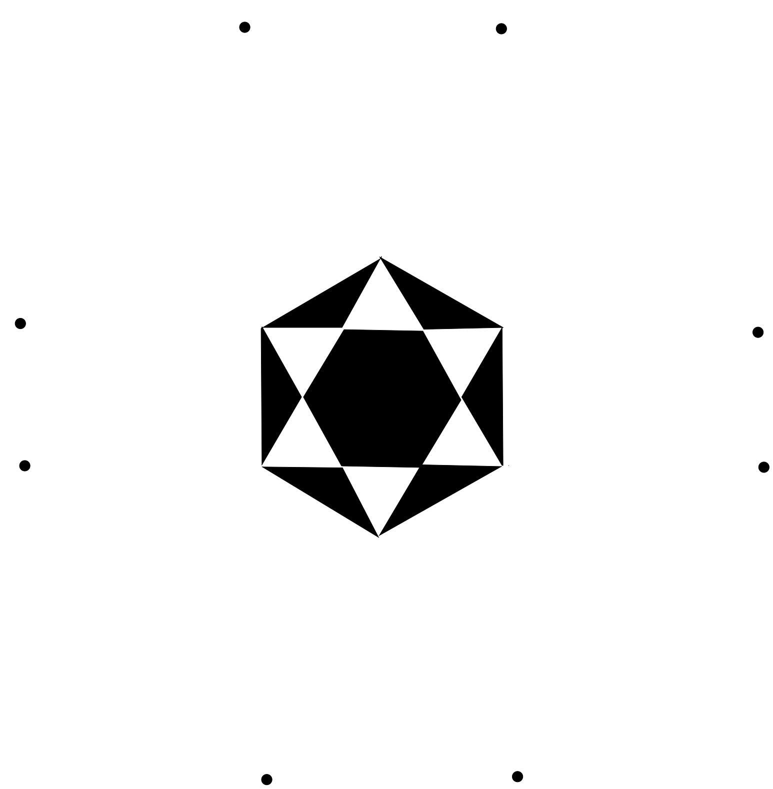
The $(1,1)$ entry of $A_\pi(\nu)$ is

$(1 - x + 3(x+y) - 3(-2x - y + 3(x+y)))(1 + 3x + y - 6(x+y) +$
$3(-2x - y + 3(x+y)))((1 + x - 3(x+y))(1 - 2x - y + 3(x+y))(3x(x+y) +$
$(1-x)(1+x+y)) + 3(-2x - y + 3(x+y))(x(1-x-y)(1 - x + 3(x+y)) +$
$(-x + 3(x+y))(3x(x+y) + (1-x)(1+x+y)))) + 3(3x + y - 6(x+y)$
$+ 3(-2x - y + 3(x+y)))((1 + 2x + y - 3(x+y))(1 + x - 3(x+y) +$
$3(-2x - y + 3(x+y)))(x(1-x-y)(1 - x + 3(x+y)) + (-x + 3(x+y))$
$(3x(x+y) + (1-x)(1+x+y))) + (x - 3(x+y) + 3(-2x - y + 3(x+y)))$
$((1 + x - 3(x+y))(1 - 2x - y + 3(x+y))(3x(x+y) + (1-x)(1+x+y)) +$
$3(-2x - y + 3(x+y))(x(1-x-y)(1 - x + 3(x+y)) +$
$(-x + 3(x+y))(3x(x+y) + (1-x)(1+x+y)))))$

The matrix $A_\pi(1/4, 1/6)$ is

$$\begin{pmatrix} \frac{207393-81191\sqrt{3}}{373248} & \frac{425}{1728} & 0 & 0 \\ \frac{425}{1728} & \frac{207393+81191\sqrt{3}}{124416} & 0 & 0 \\ 0 & 0 & \frac{167025+39913\sqrt{3}}{124416} & \frac{13}{54} \\ 0 & 0 & \frac{13}{54} & \frac{167025-39913\sqrt{3}}{373248} \end{pmatrix}$$

And the final picture is:



This is the picture of the spherical unitary representations of $G_2(\mathbb{R})$ and also $G_2(F)$ for F a p–adic field.

Example: F_4, π = the reflection representation, $\nu = (7/12, 7/24, 7/24, 0)$

$A: \{\{17908442222350428931801827\,1/27262293279626489757696,$
$28487410167323945355814 73/10097145659120922132 48,$
$43933519598549036670413563/27262293279626489757696,$
$20482625885253000050477 21/30291436977362766397 44\},$
$\{28487410167323945355814 73/10097145659120922132 48,$
$15887074726753804177812 23/10097145659120922132 48,$
$21072748454368017830220 07/30291436977362766397 44,$
$80542089182496044214547 1/30291436977362766397 44\},$
$\{43933519598549036670413563/27262293279626489757696,$
$21072748454368017830220 07/30291436977362766397 44,$
$10953541816452899251348189/27262293279626489757696,$
$16963596900691019091522 1/10097145659120922132 48\},$
$\{20482625885253000050477 21/30291436977362766397 44,$
$80542089182496044214547 1/30291436977362766397 44,$
$16963596900691019091522 1/10097145659120922132 48,$
$21851103730694390881703 9/30291436977362766397 44\}\}$

9 Computations

Here are the ingredients for computing unitarity of spherical representations.

Note: All computations must be done with rational numbers. The reason is we need to know whether a number (for example an eigenvalue) is > 0. If you compute with a fixed precision you need to be very careful you don't make a mistake.

Models of representations of Weyl groups Find a model for every irreducible representation of an exceptional Weyl group. The character tables are known.

$W(G_2)$ has 6 irreducible representations, of dimension $2, 2, 1, 1, 1, 1$.

$W(F_4)$ has 25 representations, the largest of dimension 16.

$W(E_6)$ has 25 representation, the largest of dimension 81.

$W(E_7)$ has 60 representation, the largest of dimension 512.

$W(E_8)$ has 112 representation, the largest of dimension 5600.

The facets Given a Weyl group, list the facets given by the hyperplanes $<\nu, \alpha>= 1$. Give a sample point on each facet. This is a problem in linear programming.

There are 712 facets for E_6.

Computation of $A_{\pi\nu}$ This is straightforward, but slow for large representations.

Compute whether $A_\pi(\nu)$ **is positive semi–definite** Algorithms are known, but this may be difficult for large matrices. You certainly can't compute eigenvalues. I don't know if it is possible to test a matrix of size 5600. It would be interesting to find out.

Compute the unitary set Compute the facets for which $A_{\pi,\nu}$ is positive semi–definite for all ν.

Compute a minimal set of π We should only need a small set of π to give the set of unitary ν. Find a small set of π which works.

10 Software

Here is some of the mathematical software we have available, with an emphasis on things useful for this project or similar one.

- Mathematica, Matlab, Maple: general purpose mathematical software, including symbolic computations
- Magma: Powerful algebra package, including finite groups of Lie type, Weyl groups, and character tables. Command: magma2.9 (new version of magma).
- Gap: Group Algebra Package, with an emphasis on group theory. Includes the character tables from the Atlas of Finite Groups
- LiE: Computations in Lie theory, including finite dimensional representations of complex Lie groups and algebras, Kazhdan–Lusztig polynomials
- Cocoa: Commutative Algebra package (Grobner bases, ideals, polynomials,...)
- Magnus: Combinatorial Group Theory, with a graphical interface. Handles infinite groups
- Pari: Advanced programmable calculator: symbolic computations, number theoretic functions (elliptic curves, class field theory...). Command: gp.

- Perl: Nice software for gluing things together, for example for taking the output of LiE and peparing it for input to Magma