Computer Computations in Representation Theory II: Root Systems and Weyl Groups

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6 $GL(n, \mathbb{R})$

Recall $G = GL(n, \mathbb{R})$ is the group of $n \times n$ invertible matrices over $\mathbb{R}$.n

Let $V = M_{n\times n}(\mathbb{R}) \simeq \mathbb{R}^{n^2}$ (as a vector space), and define a representation of G on V by

$$\pi(g)(X) = gXg^{-1} \quad (g \in G, X \in V)$$

The subspace of matrices of trace 0 is an irreducible representation of G.

Now let $T \subset G$ be the subgroup of diagonal matrices. Then V is a representation of T by restriction:

$$\pi(t)(X) = tXt^{-1} \quad (t \in T, X \in V)$$

Problem: Decompose V, as a representation of T, into a direct sum of irreducible representations.

Note that T is abelian; in fact $T \simeq \mathbb{R}^{*n}$. Here are some one-dimensional representations of T.

Let

$$\mathbb{Z}^n \ni \vec{k} = (k_1, \ldots, k_n) : T \to \mathbb{R}^*$$

be the map

$$diag(x_1, x_2, \ldots, x_n) \to x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$$

This is a group homomorphism.

Let $E_{i,j}$ be the matrix with a 1 in the i^{th} row and j^{th} column, and 0 elsewhere. Let $t = diag(x_1, \ldots, x_n)$. Then

$$\pi(t)E_{i,j} = \frac{x_i}{x_j} E_{i,j}$$

That is $\mathbb{C} < E_{i,j} >$ is a one-dimensional representation of T given by

$$\vec{k} = \begin{cases} (0, \ldots, 0) & i = j \\ (0, \ldots, 0, 1_i, 0, \ldots, 0, -1_j, 0, \ldots, 0) & i \neq j \end{cases}$$

with 1 in the i^{th} place and -1 in the j^{th} place.

Write $e_1, \ldots, e_n$ for the standard basis of $\mathbb{R}^n$ (or $\mathbb{Z}^n$). Then $(i \neq j)$

$$\pi(t)E_{i,j} = (e_i - e_j)(t)E_{i,j}$$

This gives the solution to the Problem: V is the direct sum of one-dimensional representations

$$e_i - e_j \quad (1 \leq i \neq j \leq n)$$

and the trivial representation $\vec{0}$ with multiplicity n.

The set

$$R = \{e_i - e_j \,|\, 1 \leq i \neq j \leq n\} \subset \mathbb{R}^n$$

is an example of a **root system**.

Now let

$$W = Norm_G(T)/T$$

For example any permutation matrix is contained in $Norm_G(T)$, and acts on T (by conjugation) by the natural permutation action. For example

$$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}x_1&&\\&x_2&\\&&x_3\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}^{-1} = \begin{pmatrix}x_2&&\\&x_3&\\&&x_1\end{pmatrix}$$

In fact $W \simeq S_n$. This is an example of a **Weyl group**.

Note that the action of W on T by conjugation gives an action of W on R, again by the natural permutation action.

6.1 Other Groups

Now let G be $Sp(2n, \mathbb{R})$ or $SO(n, \mathbb{R})$. Recall

$$G = \{g \in GL(m, \mathbb{R}) \mid gJg^t = J\}$$

with J as in Lecture I.

Let

$$\mathfrak{g} = \{X \in M_{m\times m}(\mathbb{R}) \mid XJ + JX^t = 0\}$$

The $\mathfrak{g}$ is a representation of G by

$$\pi(g)(X) = gXg^{-1} \quad (g \in G, X \in \mathfrak{g})$$

(check this).

Let T be the diagonal subgroup:

$$T = \{diag(x_1, \ldots, x_n, \frac{1}{x_1}, \ldots, \frac{1}{x_n})$$

(with an additional 1 in the case of $SO(2n+1)$).

Again T is isomorphic to $\mathbb{R}^{*n}$, and we have the one–dimensional representations $\vec{k} \in \mathbb{Z}^n$ of T. As before, decompose $\mathfrak{g}$ as a representation of T.

We get $\{e_i - e_j \,|\, 1 \leq i \neq j \leq k\}$ as before.

Let

$$F_{i,j} = \begin{pmatrix}
0 & & & & & & & & & \\
 & & & & & 0 & & & & \\
 & 0 & & & & & & & & \\
 & & & & & & & & 1_{i,j} & \\
 & & \ddots & & & & & & & \\
 & & & & & & 1_{j,i} & & & \\
 & & & 0 & & & & & & \\
 & & & & & & & & 0 & \\
 & & & & 0 & & & & & \\
0 & & & & & 0 & & & & \\
 & 0 & & & & & 0 & & & \\
 & & \ddots & & & & & \ddots & & \\
 & & & 0 & & & & & 0 & \\
 & & & & 0 & & & & & 0
\end{pmatrix}$$

Then

$$\pi(t)F_{i,j} = (e_i + e_j)(t)F_{i,j}$$

(if $i = j$ this is $2e_i(t)F_{i,i}$).

Let $R \subset \mathbb{Z}^n$ be the non-zero elements which occur. We define W as before

$$W = Norm_G(T)/T$$

This is a finite group.

$GL(n, \mathbb{R})$:

$$R = \{\pm(e_i - e_j) \,|\, 1 \leq i < j \leq n\}$$

$W \simeq S_n$ consists of all permutations in n coordinates

$Sp(2n, \mathbb{R})$:

$$R = \{\pm e_i \pm e_j \,|\, 1 \leq i < j \leq n\} \cup \{2e_i \,|\, 1 \leq i \leq n\}$$

W consists of all permutations and sign changes in n coordinates.

$SO(2n, \mathbb{R})$:

$$R = \{\pm e_i \pm e_j \,|\, 1 \leq i < j \leq n\}$$

W consists of all permutations and an even number sign changes in n coordinates.

$SO(2n + 1, \mathbb{R})$:

$$R = \{\pm e_i \pm e_j \,|\, 1 \leq i < j \leq n\} \cup \{e_i \,|\, 1 \leq i \leq n\}$$

W consists of all permutations and sign changes in n coordinates.

6.2 Formal Definition of Root Systems

Basic references are [1] and [2]. A root system R is a finite subset of $V = \mathbb{R}^n$ with the following properties. Write $(v, w) = v \cdot w$ for the standard inner product on V.

Let σ_v be the reflection in the plane orthogonal to v. Then

$$\sigma_v(w) = w - < w, v > v$$

where

$$< w, v >= 2(w, v)/(v, v)$$

The requirements are:

1. $0 \notin R$ and R spans V
2. if $\alpha \in R$ then $\pm\alpha$ are the only multiples of α in R
3. If $\alpha, \beta \in R$ then $< v, w > \in \mathbb{Z}$
4. If $\alpha \in R$ then $\sigma_\alpha : R \to R$

That is $\alpha, \beta \in R$ implies $\beta - < \beta, \alpha > \alpha \in R$.

Given R let W be the group generated by the reflections

$$\{\sigma_\alpha \,|\, \alpha \in R\}.$$

Thus W acts on R.

In the case of a root system coming from a group the Weyl group is isomorphic to $Norm_G(T)/T$.

Example: In the root system of $GL(n, \mathbb{R})$, $Sp(2n, \mathbb{R})$ or $SO(2n+1, \mathbb{R})$, $\alpha = e_i - e_j$ gives the transposition $(i\, j)$ in S^n. These generate S_n. In the case of $GL(n, \mathbb{R})$ this is all of W.

Example: In the case of $Sp(2n, \mathbb{R})$ or $SO(2n+1, \mathbb{R})$, if $\alpha = e_i$ or $2e_i$ then σ_α is the sign change in the i^{th} coordinate. These generate all permutations and sign changes, i.e. the Weyl group of type B_n.

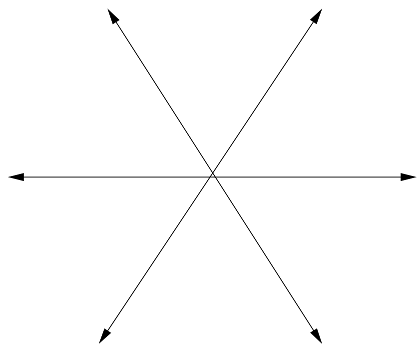
Now root systems are very rigid. In fact the possible angles between roots are $2\pi/n$ with $n = 1, 2, 3, 4, 6$. Note that $2\pi/5$ is not allowed.

Theorem 6.1 *The irreducible root systems are: A_n, B_n, C_n, D_n ($n \geq 1$) and E_6, E_7, E_8, F_4 and G_2.*

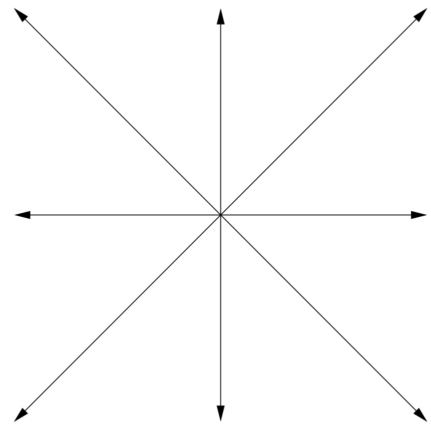
The root systems of $GL(n, \mathbb{R})$, $SO(2n+1, \mathbb{R})$, $Sp(2n, \mathbb{R})$ and $SO(2n, \mathbb{R})$ are the "classical" root systems of type A_{n-1}, B_n, C_n and D_n, respectively.

The root systems E_6, E_7, E_8, F_4 and G_2 are called the *exceptional* root systems.

Rank 2 root systems



A2



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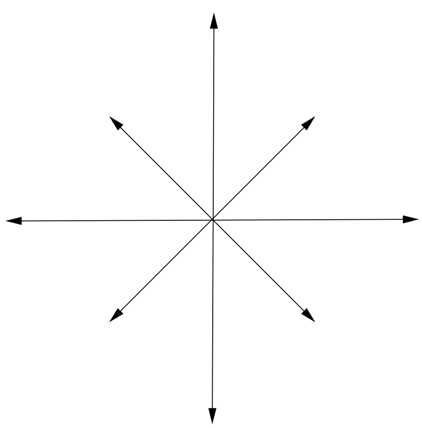
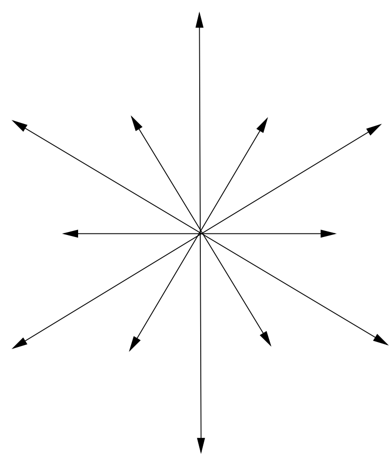


C2



G2

The Weyl groups are

$$W(A_1) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$
$$W(B_2) \simeq W(C_2) \simeq D_4$$
$$W(G_2) \simeq D_6$$

The exceptional root systems and Weyl groups

$E_8 : \pm e_i \pm e_j,\ 1 \le i < j \le 8$
$\frac{1}{2}(\epsilon_1, \ldots, \epsilon_8),\ \epsilon_i = \pm 1,\ \prod_i \epsilon_i = 1$

F_4: $\pm e_i \pm e_j,\ 1 \le i < j \le 4$
$\frac{1}{2}(\epsilon_1, \ldots, \epsilon_4),\ \epsilon_i = \pm 1$

G_2: $\pm e_i \pm e_j,\ 1 \le i < j \le 3$
$\pm(2, -1 - 1), \pm(-1, 2, -1), \pm(-1 - 1, 2)$

Type	$\lvert R\rvert$	Order(W)	realization
E_6	72	51,840	$O(6, \mathbb{F}_2)$
E_7	126	2,903,040	$O(7, \mathbb{F}_2) \times \mathbb{Z}/2\mathbb{Z}$
E_8	240	696,729,600	$W \xrightarrow{2} O(8, \mathbb{F}_2)$
F_4	48	1152	
G_2	12	12	D_6

6.3 Lie groups and root systems

Recall the groups $GL(n, \mathbb{R})$, $Sp(2n, \mathbb{R})$ and $SO(m, \mathbb{R})$ each give rise to a root system and its Weyl group. The converse holds:

Theorem 6.2 (Fantastic Theorem:) *Let R be a root system. Then there is a Lie group G for which this is the root system.*

More precisely:

- There is a subgroup G of some $GL(m, \mathbb{R})$, and
- a subspace $\mathfrak{g}$ of $M_{m\times m}(\mathbb{R})$, such that
- G acts on $\mathfrak{g}$ by $\pi(g)(X) = gXg^{-1}$
- The diagonal subgroup T is isomorphic to $\mathbb{R}^{*n}$
- The one–dimensional representations of T on $\mathfrak{g}$ are the root system $R \subset \mathbb{Z}^n \subset \mathbb{R}^n$
- The Weyl group of R is isomorphic to $Norm_G(T)$

The Lie groups of type E_6, E_7, E_8, F_4 and G_2 are among the most fascinating objects in mathematics.

Example: $E_8(\mathbb{F}_2)$ The preceding construction works over any field (this is one of the remarkable things about it). The group $E_8(\mathbb{F}_2)$ is a finite simple group of order

$$337,804,753,143,634,806,261,388,190,614,085,595,079,991,$$
$$692,242,467,651,576,160,959,909,068,800,000 \simeq 10^{75}$$

7 More on Root Systems

Let R be a root system of rank n, i.e. the ambient vector space is of dimension n. Then there is a basis $\alpha_1, \ldots, \alpha_n \in R$ of V.

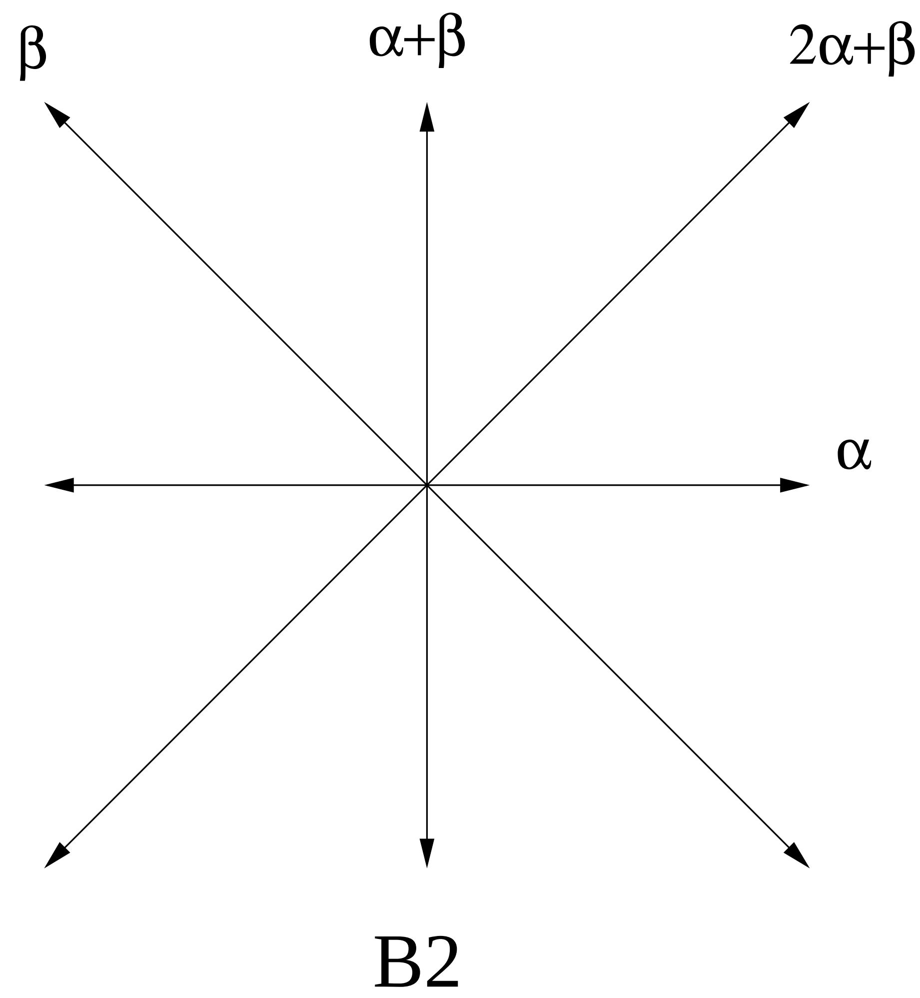
For example if R is of type A_{n-1} then we can take $\alpha_i = e_i - e_{i+1}$. The corresponding reflections are the transpositions $(i, i+1)$ in S_n, which generate S_n. This is an example of a basis with further nice properties: a set of "simple" roots.

Definition 7.1 *A set $\alpha_1, \ldots, \alpha_n$ is a set of simple roots if it is a basis of V and every root $\beta \in R$ can be written*

$$\beta = \sum_i a_i \alpha_i$$

with all $a_i \geq 0$ or all $a_i \leq 0$.

Example: $\mathbf{B_2}$



Given a set of simple roots $S = \{\alpha_1, \ldots, \alpha_n\}$ let $s_i = s_{\alpha_i}$. For $w \in W$ let $length(w)$ be the minimum k so that

$$w = s_{i_1} s_{i_2} \ldots s_{i_k}$$

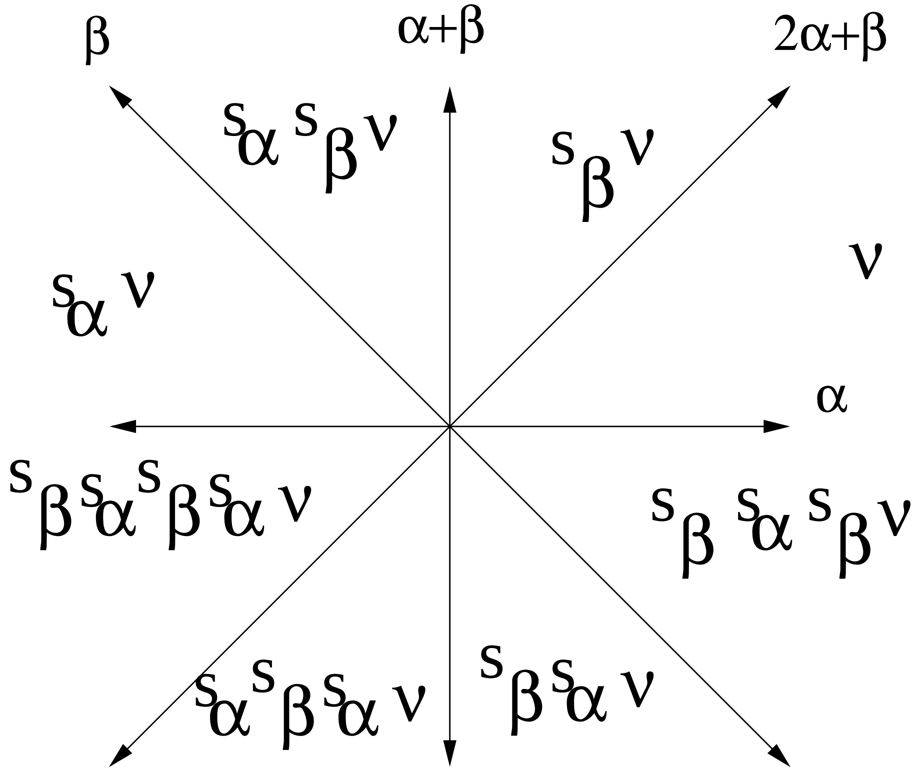
Theorem 7.2 *Let $\alpha_1, \ldots, \alpha_n$ be a set of simple roots.*

- *W is generated by $\{s_i = s_{\alpha_i} \,|\, 1 \leq i \leq n\}$.*
- *There is a unique longest element w_0 of the Weyl group*

Note: In types $B_n, C_n, D_{2n}, E_7, E_8, F_4$ and G_2, $w_0 = -I$.

Weyl group of type B_2

$$W : \{1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, s_\alpha s_\beta s_\alpha, s_\beta s_\alpha s_\beta, s_\alpha s_\beta s_\alpha s_\beta\}$$



B2

7.1 Hyperplanes

Let $R \subset V = \mathbb{R}^n$ be a root system. Recall for $\alpha, \beta \in R$,

$$< \alpha, \beta >= 2(\alpha, \beta)/(\beta, \beta)$$

where $(\alpha, \beta) = \alpha \cdot \beta$. This makes sense for any $v \in V$:

Definition 7.3 *For* $\alpha \in R, v \in V$,

$$< v, \alpha >= 2(v, \alpha)/(\alpha, \alpha)$$

Note: if there is only one root length (types A,D,E) we may take $(\alpha, \alpha) = 2$ for all $\alpha \in R$, and then

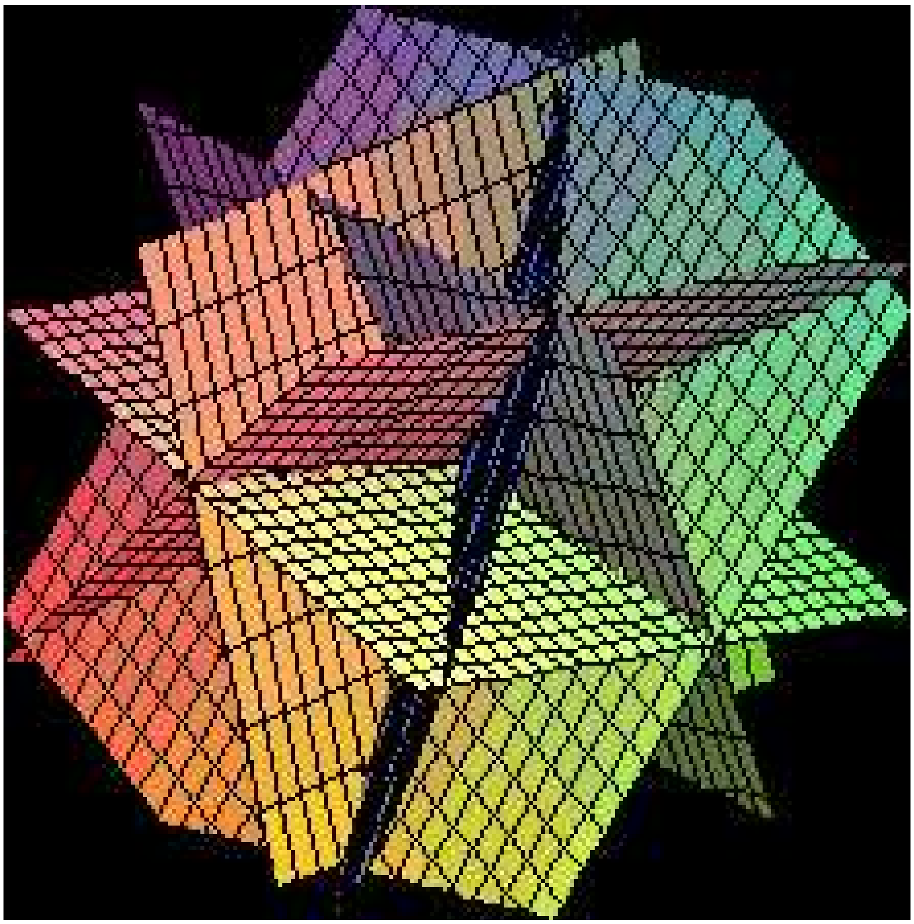
$$< v, \alpha >= (v, \alpha)$$

You may want to think about this case at first.

Now each $\alpha \in R$ gives a hyperplane

$$\{v \mid < v, \alpha >= 0\}$$

Hyperplanes of A_3

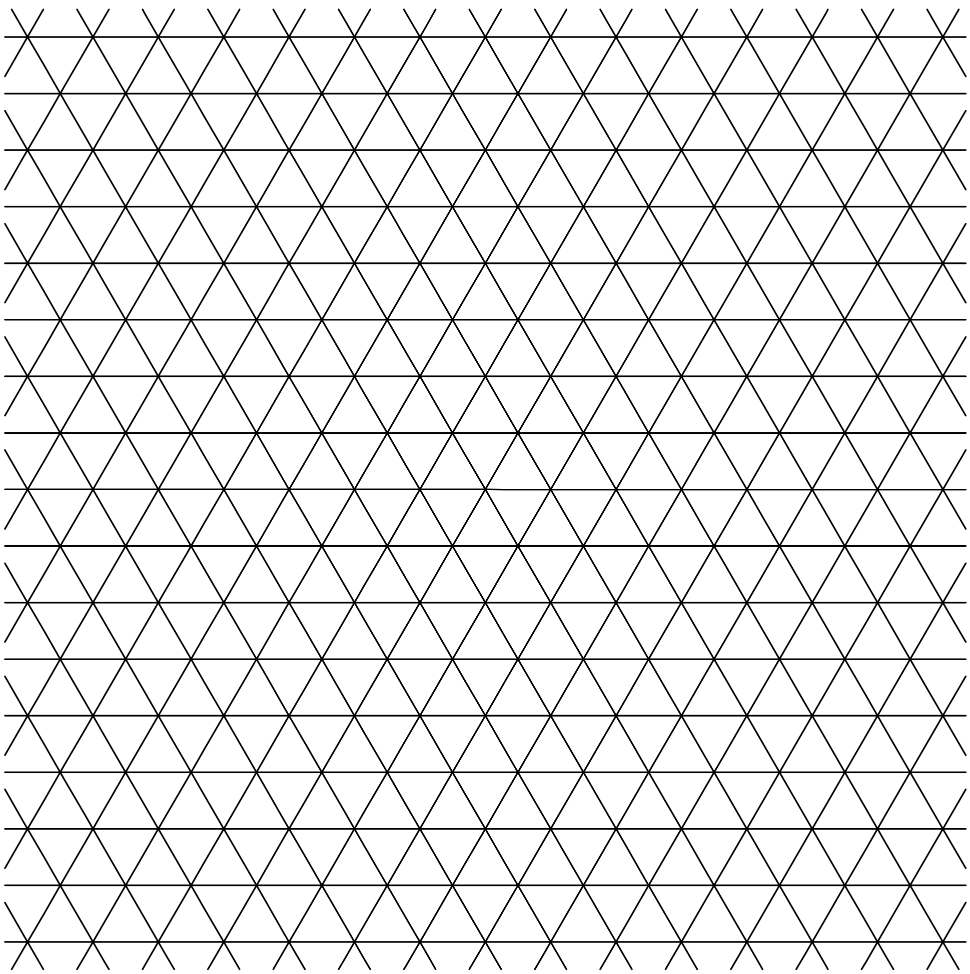


More generally each $k \in \mathbb{Z}, \alpha \in R$ gives a hyperplane

$$\{v \mid < v, \alpha >= k$$

This breaks V up into countably many facets.

A2



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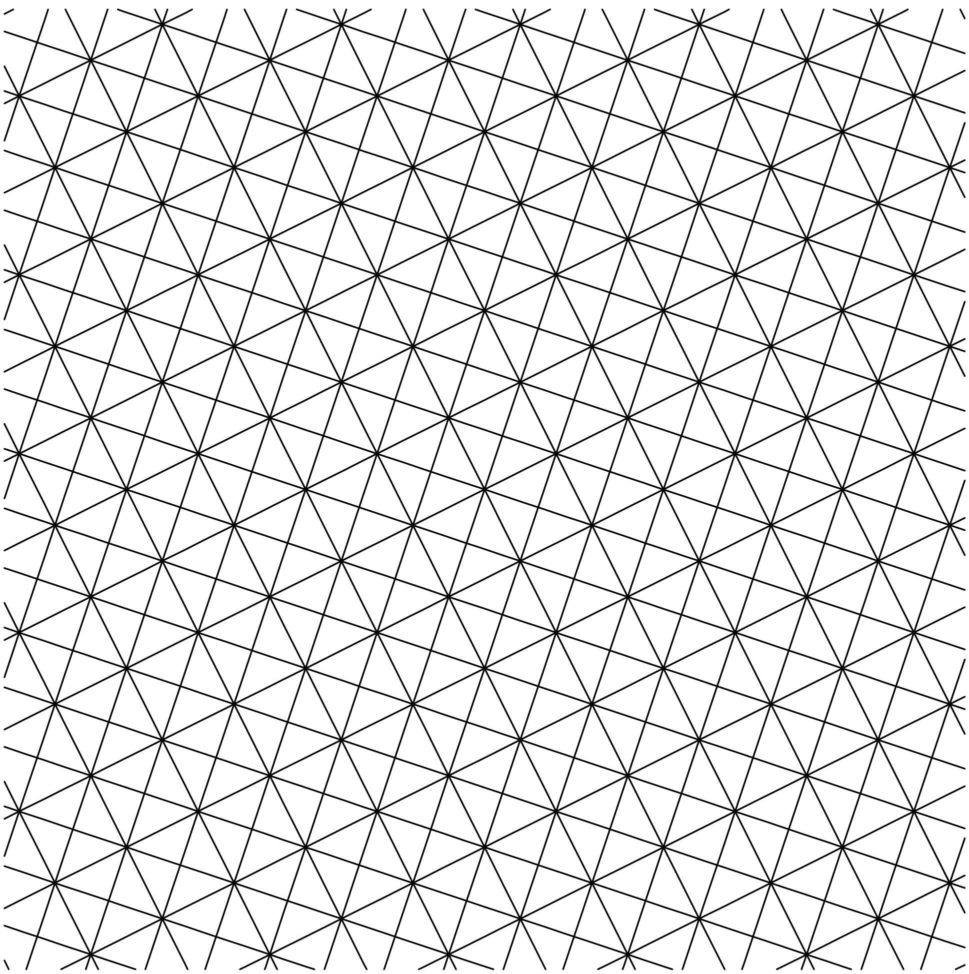

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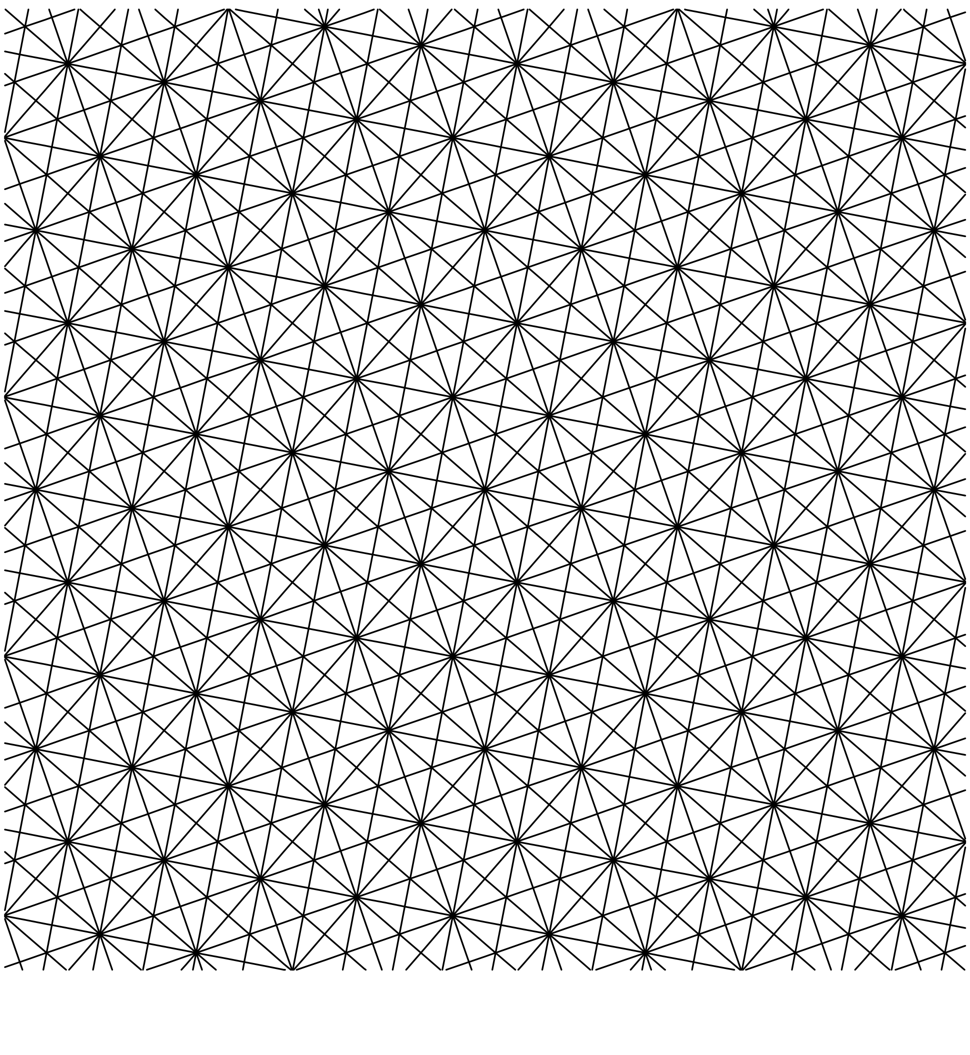

References

[1] J. Humphreys. *Introduction to Lie Algebras and Representation Theory.* Springer-Verlag, Berlin,Heidelberg,New York, 1972.

[2] James E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1990.