# Some notes on parametrizing representations

December 13, 2005

These notes are an attempt to clarify some confusing issues about parametrizing representations. This includes the basepoint issue. At least at the moment this is only of interest to the experts.

## 1 The crux of the biscuit

These notes have gotten rambling and disorganized, so here is the long and the short of it. For details see the relevant sections.

We want to go between Fokko's data and some other parametrization of representations of $G$. The most direct route is to the modules $I(\Psi^+, \Lambda)$ of [6].

The nice thing is that in the setting of [4] we fix almost everything, including $H, B$ and even $\lambda$, the differential of $\Lambda$. The only thing which takes some work is specifying $\Lambda$ off of the identity component of $H(\mathbb{R})$.

The data $(y, \lambda)$ parametrize maps of $W_{\mathbb{R}}$ into an E-group for a torus, but an additional choice is necessary (the distinguished element of this E-group) to actually get a representation of (a cover of) a torus. This is the basepoint question.

Many choices of basepoints will give a bijection between Fokko's data and representations of strong real forms of $G$. These choices are pinned down by requiring that cross actions and Cayley transforms of representations correspond to the natural cross action and Cayley transform defined in [4].

Note that compatibility with cross action includes the fact that the gradings/cogradings of imaginary and compact roots behave correctly.

Compatibility with cross action primarily says that the basepoints we choose, i.e. certain elements $y_0$ on the dual side, should be large. Furthermore compatibility with Cayley transforms says they should be obtained from a fixed $\delta^\vee$ by a particular series of Cayley transforms. We spell the choices out in Section 5. There is a conjecture there, verified by Fokko by computer, about this issue (Conjecture 5.19). The work to show these choices are correct is in Section 6.

The bottom line is this: always use our fixed $B$ and its corresponding set $\mathbf{\Psi}^+$ of positive roots. For a positive system of real roots use $\mathbf{\Psi}^+_{R,x}$, the roots in $\mathbf{\Psi}^+$ which are real with respect to $\theta_x$. Finally use the basepoints provided by Fokko's algorithm. (These last two choices are related, you can modify them both simultaneously by $W_{im}$). Then you'll get the correct parametrization.

## 1.1 Some Objects

It might be useful to collect some objects here which are defined later.

Our group is $G$, and we have fixed $H$ and $B$. The parameter set is $\mathcal{Z}$, and $(x, y)$ is a typical element of $\mathcal{Z}$.

An element $x$ defines a real form $H_x(\mathbb{R})$ of $H$. A typical character of a cover of $H_x(\mathbb{R})$ is written $\Lambda$. We parametrize characters $\Lambda$ as

$$\Lambda(x, \lambda, \nu)$$

(cf. Lemma 2.1).

We define the cross action

$$w \times \Lambda = \Lambda \otimes \mu$$

with $\mu = w\lambda - \lambda$ (Definition 2.14). This may also be written

$$w \times \Lambda(x, \lambda, \nu) = \Lambda(x, \lambda + \mu, \nu + \mu).$$

The Cayley transform is

$$c^\alpha \Lambda(x, \lambda, \nu) = \{\Lambda(\sigma_\alpha x, \lambda, \nu), \Lambda(\sigma_\alpha x, \lambda, \nu + \alpha)\}$$

(Definition 2.19).

We parametrize characters $\Lambda$ in terms of maps of $W_\mathbb{R}$ into the dual group as

$$\Lambda[x, y, \lambda, y_0]$$

(3.4)

We then compute the cross and Cayley transforms in these coordinates:

$$w \times \Lambda(x, y, \lambda, y_0) = \Lambda(x, y, w\lambda, y_0)$$

(Lemma 3.7) and

$$c^\alpha \Lambda(x, y, \lambda, y_0) = \Lambda(\sigma_\alpha x, \sigma^\alpha y, \lambda, \sigma^\alpha y_0)$$

(Lemma 3.12).

A standard module is $I(x, \Psi_R^+, \Lambda)$ (Definition 6.2). Here $x$ specifies the (strong) real form, $\Psi_R^+$ is a set of positive real roots with respect to $\theta_x$, and $\Lambda$ is a genuine character of a cover of $H_x(\mathbb{R})$. We may write $\Psi_{R,x}^+$ to indicate that the real roots are defined by $x$. The differential of $\Lambda$ is $\lambda$, which is fixed, and is the infinitesimal character. Usually we can take $\Psi_R^+ = \boldsymbol{\Psi}_R^+$, and the standard module is

$$I(x, \boldsymbol{\Psi}_R^+, \Lambda)$$

and $(\boldsymbol{\Psi}_R^+, \Lambda)$ are in good position (Definition 6.1).

The cross action is given by (cf. Remark 1.1)

$$w \times I(x, \boldsymbol{\Psi}_R^+, \Lambda) = I(x, \boldsymbol{\Psi}_R^+, w^{-1} \times \Lambda)$$

(Lemma 6.13) and Cayley transforms by

$$c^\alpha I(x, \Psi_{R,x}^+, \Lambda) = I(\sigma_\alpha x, \Psi_{R,\sigma_\alpha x}^+, c^\alpha \Lambda)$$

(Proposition 6.21).

Finally we get to the parametrization of [4], so a standard module is

$$I(x, y, \lambda)$$

(Definition 6.7), and the actions are

$$w \times I(x, y, \lambda) = I(w \cdot x, w \cdot y, \lambda)$$

and

$$c_\alpha I(x, y, \lambda) = I(\sigma_\alpha x, \sigma^\alpha y, \lambda).$$

**Remark 1.1** The cross action is defined on parameters, not representations. The only action canonically defined on representations is coherent continuation. Hence, as David points out, the notation $w \times I(x, \Psi_R^+, \Lambda)$ is misleading (even though it appears in [5].) I don't want to fix this now.

One point is we could define $w \times I(x, \psi_R^+, \Lambda)$ to be $\pm$ the coherent continuation action of $w$ on this representation, modulo representations coming from less compact Cartan subgroups.

## 2 Covers of Tori

Suppose $H$ is a complex algebraic torus and $\gamma \in \frac{1}{2}X^*(H)$. Then the two-fold cover $H_\gamma$ is defined. Suppose $\theta$ is a holomorphic involution of $H$. Then $H_{\theta\gamma}$ is defined. We assume $\theta\gamma - \gamma \in X^*(H)$, so that $H_{\theta\gamma}$ is canonically isomorphic to $H_\gamma$. (In our applications we will have $\gamma = \rho$.) Then $\theta$ acts on $H_\gamma$ and we can talk about $(\mathfrak{h}, H_\gamma^\theta)$-modules. These correspond to representations of a real form of a two-fold cover $H_\gamma(\mathbb{R})$ of $H$. See [6, Proposition 5.8] and [7].

**Lemma 2.1** *Fix $\gamma \in X^*(H) \otimes \mathbb{C}$ and $\theta$ satisfying $\theta\gamma - \gamma \in X^*(H)$. Write $H_\gamma(\mathbb{R})$ for the corresponding real form of $H_\gamma$. Suppose $(\lambda, \nu)$ satisfy $\lambda \in X^*(H) \otimes \mathbb{C}$, $\nu \in \gamma + X^*(H)$ and*

(2.2)(a) $$\lambda + \theta\lambda = \nu + \theta\nu.$$

*Then we obtain a genuine $H_\gamma(\mathbb{R})$-module $\Lambda(\lambda, \nu)$ with differential $\lambda$.*

*Furthermore $\Lambda(\lambda, \nu) = \Lambda(\lambda', \nu')$ if and only if $\lambda = \lambda'$ and*

(2.2)(b) $$\nu' - \nu \in (1 - \theta)X^*(H).$$

*If $\gamma \in X^*(H)$ we recover representations of $H(\mathbb{R})$ as in [6, Proposition 3.26].*

**Remark 2.3** The dual Cartan involution is $\theta^\vee = -\theta$, and we can write the Lemma in terms of $\theta^\vee$ instead: $\Lambda(x, \lambda, \nu)$ is defined if

(2.4)(a) $$\lambda - \theta^\vee\lambda = \nu - \theta^\vee\nu$$

and two are equal if

(2.4)(b) $$\nu' - \nu \in (1 + \theta^\vee)X^*(H).$$

The group $H_\gamma(\mathbb{R})$ comes equipped with a genuine character $\gamma$. The character $2\gamma$ factors to $H_\gamma(\mathbb{R})$. Its absolute value has a unique positive square root, which we denote $|\gamma|$. Then $\gamma/|\gamma|$ is naturally a genuine representation of $H_\gamma(\mathbb{R})$. [Check: it doesn't matter, as in [6, Example 8.13], that $2\gamma$ is not real valued?]

**Lemma 2.5** *1. $\gamma = \Lambda(\gamma, \gamma)$.*

*2.* $|\gamma| = \Lambda(\frac{1}{2}(\gamma + \overline{\gamma}), 0)$.

*3.* $\gamma/|\gamma| = \Lambda(\frac{1}{2}(\gamma - \overline{\gamma}), \gamma)$

**Remark 2.6** In most applications we will have $\overline{\gamma} = -\theta\gamma$ (since $2\gamma$ is in the root lattice) so these become:

1. $\gamma = \Lambda(\gamma, \gamma)$.
2. $|\gamma| = \Lambda(\frac{1}{2}(\gamma - \theta\gamma), 0)$.
3. $\gamma/|\gamma| = \Lambda(\frac{1}{2}(\gamma + \theta\gamma), \gamma)$

In fact we'll often have $\overline{\gamma} = \gamma$, i.e. $\theta\gamma = -\theta\gamma$, in which case these become

1. $\gamma = \Lambda(\gamma, \gamma)$.
2. $|\gamma| = \Lambda(\gamma, 0)$.
3. $\gamma/|\gamma| = \Lambda(0, \gamma)$

**Remark 2.7** Check these assertions.

For later use we do a few calculations in these coordinates. We suppose $H$ is a $\theta$-stable Cartan subgroup of $G$.

First of all if $\alpha$ is a root then

$$\alpha = \Lambda(\alpha, \alpha)$$

and if $\alpha$ is real then $\mathrm{sgn}(\alpha) = \alpha/|\alpha|$ and so

$$\mathrm{sgn}(\alpha) = \Lambda(0, \alpha).$$

Write $\Psi$ for the roots of $H$ in $G$, and $\Psi_R$ for the real roots. Suppose $\Psi_R^+$ is a set of positive real roots and let $\rho_R = \frac{1}{2}\sum_{\alpha\in\Psi_R^+}\alpha$.

Note that $\theta(\rho_R) = -\rho_R$, and $2\rho_R$ is real valued. Take $\gamma = \rho_R$ so we have

$$\rho_R/|\rho_R| = \Lambda(0, \rho_R)$$

Now take $w \in W_R$, the Weyl group of $\Psi_R$. Recall [6, (8.26)(a)] $\tau(\Psi_R^+, w)$:

$$\tau(\Psi_R^+, w) = (\rho_R/|\rho_R|)/(w\rho_R/|w\rho_R|) \tag{2.8}$$

That is

$$\tau(\Psi_R^+, w) = \Lambda(0, \rho_R - w\rho_R)$$

In particular if $\alpha$ is a simple root of $\Psi_R^+$ we have

$$\tau(\Psi_R^+, s_\alpha) = \Lambda(0, \alpha) = \mathrm{sgn}(\alpha). \tag{2.9}$$

We may drop $\Psi_R^+$ from the notation.

## 2.1 Cross action on $\Lambda$

Now suppose $H$ is a Cartan subgroup of $G$. A critical role is played by the action of $W(G,H)$. Suppose $x$ is a strong real form of $G$, let $\theta = \theta_x$ and assumem $H$ is $\theta$ stable. Also assume $\theta\gamma - \gamma \in X^*(H)$. We talk about $(\mathfrak{h}, H_\gamma^\theta)$ modules, but we should also keep the strong real form $x$ of $H$ in mind. So we write $\Lambda(x, \lambda, \nu)$ to indicate this is a representation of the strong real form $x$ of $H$.

Suppose $w \in W = W(G,H)$. Then $w$ acts on everything in sight, and

**Definition 2.10** *Suppose* $\Lambda = \Lambda(x, \lambda, \nu)$. *Then*

$$w\Lambda = \Lambda(wx, w\lambda, w\nu) \tag{2.11}$$

This is only a representation of the same group if $wx = x$, i.e. $w \in W^\theta$. In this case we have:

**Lemma 2.12** *Suppose $w \in W^\theta$, i.e. $wx = x$. Then $w$ acts on $H_\gamma(\mathbb{R})$, therefore on genuine characters of this group, and the action just described agrees with this action.*

**Remark 2.13** Check this - it must be true...

We also define the cross action of $W$. Suppose $\Lambda$ is a character of $H_\gamma(\mathbb{R})$ and $w \in W(d\Lambda)$, i.e. $w(d\Lambda) = d\Lambda + \mu$ for $\mu \in X^*(H)$.

**Definition 2.14** *Let*

$$w \times \Lambda = \Lambda \otimes \mu$$

See [5, page 44] and [8, Definition 8.3.1]. This definition is simpler than the of [8], because it is in terms of $\Lambda$ rather then $\Gamma$.

**Lemma 2.15** *Write $w\lambda = \lambda + \mu$ with $\mu \in X^*(H)$.*

$$\begin{aligned} w \times \Lambda(x,\lambda,\nu) &= \Lambda(x,\lambda+\mu,\nu+\mu) \\ &= \Lambda(x,w\lambda,\nu+\mu) \\ &= \Lambda(x,w\lambda,\nu+(w\lambda-\lambda)) \end{aligned}$$

*In particular*

$$w \times \Lambda(x,\lambda,\lambda) = \Lambda(x,w\lambda,w\lambda)$$

We emphasize the difference between $w\Lambda$ and $w \times \Lambda$:

$$\begin{aligned} w\Lambda(x,\lambda,\nu) &= \Lambda(wx,w\lambda,w\nu) \\ w \times \Lambda(x,\lambda,\nu) &= \Lambda(x,w\lambda,\nu+(w\lambda-\lambda). \end{aligned}$$

In particular

$$\begin{aligned} w^{-1}(w \times \Lambda(x,\lambda,\nu)) &= \Lambda(w^{-1}x,\lambda,w^{-1}(\nu+\mu)) \\ &= \Lambda(w^{-1}x,\lambda,w^{-1}\nu+(\lambda-w^{-1}\lambda)) \\ &= \Lambda(w^{-1}x,\lambda,\nu+(\lambda-\nu)-w^{-1}(\lambda-\nu)) \end{aligned}$$

This latter operation is relevant since it gives us a character with the same differential, albeit possibly for a different group.

Finally suppose $\alpha$ is a root and set $m = \langle\lambda,\alpha^\vee\rangle, n = \langle\nu,\alpha^\vee\rangle$. Then

$$\begin{aligned} s_\alpha\Lambda(x,\lambda,\nu) &= \Lambda(s_\alpha x,\lambda-m\alpha,\nu-n\alpha) \\ s_\alpha \times \Lambda(x,\lambda,\nu) &= \Lambda(x,\lambda-m\alpha,\nu-m\alpha) \\ s_\alpha(s_\alpha \times \Lambda(x,\lambda,\nu)) &= \Lambda(s_\alpha x,\lambda,\nu+(m-n)\alpha) \end{aligned} \tag{2.16}$$

An important special case is $w = s_\alpha$ for $\alpha$ a real root:

$$s_\alpha(s_\alpha \times \Lambda(x,\lambda,\nu)) = \Lambda(x,\lambda,\nu+(\lambda-\nu)-s_\alpha(\lambda-\nu)) \tag{2.17}$$

Recall $\tau(s_\alpha) = \Lambda(0,\rho_R - s_\alpha\rho_R)$. For later use we note

(2.18)

$$\begin{aligned} s_\alpha(s_\alpha \times \Lambda(x,\lambda,\nu)) \otimes \tau(s_\alpha) &= \Lambda(x,\lambda,\nu+(\lambda-\nu+\rho_R)-s_\alpha(\lambda-\nu+\rho_R)) \\ &= \Lambda(x,\lambda,\nu) \otimes \Lambda(0,\langle\lambda-\nu+\rho_R,\alpha^\vee\rangle\alpha) \\ &= \Lambda(x,\lambda,\nu) \otimes \mathrm{sgn}(\alpha)^{\langle\lambda-\nu+\rho_R,\alpha^\vee\rangle} \end{aligned}$$

## 2.2 Cayley transform of $\Lambda$

We now turn to Cayley transforms.

Suppose $\Lambda = \Lambda(x, \lambda, \nu)$, of course $(\lambda, \nu)$ satisfy (2.2)(a) and $\theta = \theta_x$. Suppose $\alpha$ is a non-compact imaginary root for $\theta$.

**Definition 2.19**

$$c^\alpha \Lambda(x, \lambda, \nu) = \{\Lambda(\sigma_\alpha x, \lambda, \nu), \Lambda(\sigma_\alpha x, \lambda, \nu + \alpha)\} \tag{2.20}$$

Here $\sigma_\alpha$ is as in [4]. We check the characters on the right hand side are well defined. We have $\theta_{c_\alpha x} = s_\alpha \theta$.

$$\begin{aligned}(\lambda + s_\alpha\theta\lambda) - (\nu + s_\alpha\theta\nu) &= (\lambda + \theta\lambda - \langle\theta\lambda, \alpha^\vee\rangle) - (\nu + \theta\nu - \langle\theta\nu, \alpha^\vee\rangle\alpha \\ &= (\lambda + \theta\lambda) - (\nu + \theta\nu) + \langle\theta(\nu - \lambda), \alpha^\vee\rangle \\ &= \langle\nu - \lambda, \alpha^\vee\rangle \text{ by (2.2)(a)}\end{aligned}$$

We need to show this is 0. By (2.2)(a) we have

$$\langle(\lambda - \nu) + (\theta\lambda - \theta\nu), \alpha^\vee\rangle = 0$$

and since $\theta\alpha^\vee = \alpha^\vee$ we conclude

$$2\langle\lambda - \nu, \alpha^\vee\rangle = 0$$

Note that the two characters on the right hand side of (2.20) might be same. Note that

$$\Lambda(\sigma_\alpha x, \lambda, \nu + \alpha) = \Lambda(\sigma_\alpha x, \lambda, \nu) \otimes \mathrm{sgn}(\alpha)$$

and these are different if $\alpha : H_{c_\alpha x}(\mathbb{R}) \to \mathbb{R}^*$ is surjective, i.e. $\alpha$ (now thought of as a real root) is type $I$. Equivalently by (2.2)(b) this holds if

$$\alpha \in (1 - \theta)X^*(H).$$

For example if $G = SL(2)$ and $\alpha$ is a real root then $\alpha = (1 - \theta)\alpha/2$, so these are the same. If $G = PGL(2)$ then $\alpha/2 \notin X^*(H)$ and these characters are distinct.

# 3 Maps of the Weil group

Fix $y$. The data $(y,\lambda)$ defines a map $\phi : W_{\mathbb{R}} \to \langle H^\vee, y\rangle$. Choose $y_0 \in H^\vee y$, i.e. $p(y_0) = p(y)$. Then $\langle H^\vee, y\rangle = \langle H^\vee, y_0\rangle$ is an E-group for $H$; by this we mean we consider $y_0$ to be the distinguished element [6, Definition 5.9]. Recall

(3.1)(a) $$y^2 = e^{2\pi i\lambda}$$

and $\lambda \in X^*(H)$ since we are in the integral case. Choose $\gamma$ so that

(3.1)(b) $$y_0^2 = e^{2\pi i\gamma}$$

and recall $2\gamma \in X^*(H)$. Write

(3.1)(c) $$y = e^{2\pi i\tau} y_0$$

for some $\tau \in X^*(H^\vee) \otimes \mathbb{C}$. Squaring both sides and using (3.1)(a,b) we conclude

(3.1)(d) $$e^{2\pi i\gamma} = e^{2\pi i(\lambda - (\tau + \theta^\vee\tau))}$$

i.e.

(3.1)(e) $$\lambda - (\tau + \theta^\vee\tau) \in \gamma + X^*(H)$$

**Definition 3.2** *Given $(y,\lambda,y_0)$ let $\phi(y,\lambda,y_0) : W_{\mathbb{R}} \to \langle H^\vee, y_0\rangle$ be the corresponding map. That is*

(3.3) $$\begin{aligned}\phi(z) &= z^\lambda \overline{z}^{y\lambda}\\ \phi(j) &= e^{-\pi i\lambda} y\\ &= e^{2\pi i(\tau - \frac{1}{2}\lambda)} y_0\end{aligned}$$

*Suppose $(x,y) \in \mathcal{Z}$. Then $\phi$ defines a genuine character*

(3.4) $$\Lambda[x, y, \lambda, y_0]$$

*of $H_\gamma(\mathbb{R})$.*

**Lemma 3.5** *Suppose we are given* $(x,y) \in \mathcal{Z}$ *and* $(\lambda, y_0)$. *We assume* $e^{2\pi i\lambda} = y^2$ *as usual. Choose* $\tau$ *satisfying*

$$y = e^{2\pi i\tau} y_0.$$

*Then in the notion of Section 2 we have*

$$\Lambda[x, y, \lambda, y_0] = \Lambda(x, \lambda, \kappa)$$

*with*

$$\kappa = \lambda - (\tau + \theta^\vee \tau)$$

*This is a representation of* $H_\gamma(\mathbb{R})$ *where* $e^{2\pi i\gamma} = y_0^2$.

Note that by (3.1)(e) $\kappa \in \gamma + X^*(H)$ so this defines a representation of the $\gamma$ cover of $H_x(\mathbb{R})$.

**Example 3.6** For example suppose $y = y_0$, so $h = 1, \lambda = \gamma$ and $\tau = 0$. Then

$$\Lambda = \chi(x, \lambda, \lambda).$$

This is a representation of $H_x(\mathbb{R})_\lambda$, which is the trivial cover if $\lambda \in X^*(H)$. This is (the restriction of) a holomorphic character of $H_\lambda$.

**Proof.** The main point is that

$$\begin{aligned}
\phi(z) &= z^\lambda \overline{z}^{y\lambda} \\
\phi(j) &= e^{-\pi i\lambda} y \\
&= e^{-\pi i\lambda} h y_0 \\
&= e^{-\pi i\lambda + 2\pi i\tau} y_0 \\
&= e^{2\pi i(\tau - \frac{1}{2}\lambda)} y_0
\end{aligned}$$

Then by [6, (4.7)(c)] $\Lambda = \Lambda(x, \lambda, \kappa)$ with

$$\begin{aligned}
\kappa &= \frac{1}{2}(\lambda - \theta^\vee \lambda) - ((\tau - \frac{1}{2}\lambda) + \theta^\vee(\tau - \frac{1}{2}\lambda)) \\
&= \lambda - (\tau + \theta^\vee \tau)
\end{aligned}$$

$\square$

## 3.1 The Cross Action

We now have to compute the standard and cross actions of $W$ in terms of the parameters $\Lambda[x, y, \lambda, y_0]$.

**Lemma 3.7**

$$w \times \Lambda[x, y, \lambda, y_0] = \Lambda[x, y, w\lambda, y_0] \tag{3.8}$$

**Proof.** This is perhaps surprising. Let's see, write $y = e^{2\pi i\tau} y_0$, and then

$$\begin{aligned} w \times \Lambda[x, y, \lambda, y_0] &= w \times \Lambda(x, \lambda, \lambda - (\tau + \theta^\vee \tau)) \\ &= \Lambda(x, w\lambda, \lambda - (\tau + \theta^\vee \tau) + (w\lambda - \lambda)) \\ &= \Lambda(x, w\lambda, w\lambda - (\tau + \theta^\vee \tau)) \\ &= \Lambda[x, y, w\lambda, y_0] \end{aligned} \tag{3.9}$$

□

**Lemma 3.10**

$$w\Lambda[x, y, \lambda, y_0] = \Lambda[wx, wy, w\lambda, wy_0] \tag{3.11}$$

This is the natural action of $W$.

We sometimes combine the two:

$$w^{-1}(w \times \Lambda[x, y, \lambda, y_0]) = \Lambda[w^{-1}x, w^{-1}y, \lambda, w^{-1}y_0]$$

which is handy since this has the same $\lambda$.

## 3.2 Cayley Transforms

We next need to compute $c^\alpha$ in the $\Lambda[x, y, \lambda, y_0]$ coordinates.

**Lemma 3.12** *Suppose $\alpha$ is a noncompact imaginary root. Then*

$$c^\alpha \Lambda[x, y, \lambda, y_0] = \Lambda[\sigma_\alpha x, \sigma^\alpha y, \lambda, \sigma^\alpha y_0] \tag{3.13}$$

Here $\sigma^\alpha$ is the possibly multivalued inverse cayley transform. That is

$$\sigma^\alpha y = \{y' \mid \sigma_\alpha y' = y\}.$$

**Proof.** Let $\theta = \theta_x$. Write $y = e^{2\pi i\tau} y_0$, so

$$\Lambda[x, y, \lambda, y_0] = \Lambda(x, \lambda, \nu)$$

with

$$\nu = \lambda - (\tau + \theta^\vee \tau).$$

Then

$$\begin{aligned} c^\alpha \Lambda[x, y, \lambda, y_0] &= c^\alpha \Lambda(x, \lambda, \nu) \\ &= \Lambda(\sigma_\alpha x, \lambda, \nu) \end{aligned}$$

On the other hand we have

$$\begin{aligned} \sigma_\alpha y &= \sigma_\alpha e^{2\pi i\tau} y_0 \\ &= e^{2\pi i s_\alpha \tau} \sigma_\alpha y_0 \end{aligned}$$

so

$$\Lambda[\sigma_\alpha x, \sigma^\alpha y, \lambda, \sigma^\alpha y_0] = \Lambda(\sigma_\alpha x, \lambda, \nu')$$

with

$$\begin{aligned} \nu' &= \lambda - (s_\alpha \tau + s_\alpha \theta^\vee s_\alpha \tau) \\ &= \lambda - (s_\alpha \tau + \theta^\vee \tau) \end{aligned}$$

Therefore

$$\begin{aligned} \nu' - \nu &= \lambda - (s_\alpha \tau + \theta^\vee \tau) - \lambda + (\tau + \theta^\vee \tau) \\ &= \tau - s_\alpha \tau \\ &= \langle \tau, \alpha^\vee \rangle \alpha^\vee \end{aligned}$$

We're on the less compact Cartan here, i.e. $\alpha$ is a real root here, or in other words $s_\alpha \theta \alpha = -\alpha$. Therefore $\alpha^\vee = (1 - s_\alpha \theta)\frac{1}{2}\alpha^\vee$, so the result follows from (2.2)(b).

□

# 4 The Setup

We need to go back and forth a bit between the notation of Fokko's notes [4] and that of [3]. See also [1], [5].

As in [3, Section 8] we are given $(G, \mathcal{S})$. Here $G$ is a (complex) reductive algebraic group, and $\mathcal{S}$ is a conjugacy class of (distinguished) splittings of the usual exact sequence, in other words a conjugacy class of elements $\delta$ such that $\theta_\delta$ is a fundamental involution. To avoid confusion we may write such elements as $\delta_f$.

Equivalently we are given a conjugacy class $\mathcal{D}$ of pairs $(\delta, B)$ where $B$ is large with respect to $\delta$. Thus $\mathrm{int}(\delta)$ is a quasisplit involution, so we may write it $\delta_q$. Recall [6, 9.7(d)]

$$\delta_q = m_\rho \delta_f$$

Similarly on the dual side we have $G^\vee$, $\delta_f^\vee \in \mathcal{S}^\vee$, and $(\delta_q^\vee, B^\vee) \in \mathcal{D}^\vee$.

We will work in the Fokko's setting, so we will fix $\delta, \delta^\vee$ and $H, B, H^\vee, B^\vee$. Here $H$ is fundamental, and $B$ is large, with respect to $\delta = \delta_f$, and similarly on the dual side.

Recall the definitions of $\mathcal{X}, \mathcal{I}$ and $\mathcal{Z}$ [4]. In particular $\mathcal{I} \subset W\delta$ is the set of twisted involutions, and there is a map $p : \mathcal{X} \to \mathcal{I}$. We write $\sigma\cdot$ for the action of $\sigma \in W$ on $\mathcal{X}$ and $\mathcal{I}$. This map is equivariant for the actions of $W$ on $\mathcal{X}$ and $\mathcal{I}$.

Our basic data will be a pair $(x, y) \in \mathcal{Z}$. Together with an element $\lambda \in X_*(H^\vee) \otimes \mathbb{C}$, satisfying $e^{2\pi i \lambda} = y^2$, these will parametrize various things, including maps of Weil groups, characters of Cartans of various sorts, and representations. This is what we want to make precise.

An important but barely visible role is played by $\zeta$. In the setting of [4] $\zeta$ is fixed, and gives identifications

$$X^*(H) = X_*(H^\vee), \quad X_*(H) = X^*(H^\vee).$$

In particaulr if $\alpha \in X^*(H)$ is a root of $H$ in $G$ then $\alpha^\vee \in X_*(H) = X^*(H^\vee)$ is a root of $H^\vee$ in $G^\vee$. We use these implicitly. We also identify $W(G, H)$ and $W(G^\vee, H^\vee)$.

Our fixed Borel $B$ gives a fixed set $\mathbf{\Psi}^+$ (note the bold face) of positive roots.

## 4.1 Some notation about root systems

The notation is verging on incomprehensible. This is an attempt to clarify it.

We have our fixed root system $\mathbf{\Psi}$, and positive roots $\mathbf{\Psi}^+$.

If $x \in \mathcal{X}$ then $\theta_x$ is an involution of $G$. Its restriction to $H$ only depends on $p(x) \in \mathcal{I}$. We write the real roots as $\mathbf{\Psi}_{R,x}$ or $\mathbf{\Psi}_{R,w}$ to indicate this dependence, and $\mathbf{\Psi}_{im,x}\mathbf{\Psi}_{im,w}$ similarly.

We define $\mathbf{\Psi}^+_{R,x} = \mathbf{\Psi}^+ \cap \mathbf{\Psi}_{R,x}$, and other similar versions.

If $\Psi^+$ is an arbitrary set of positive roots, we define $\Psi^+_{R,x}$ etc. the obvious way.

On occasion we will talk about simple roots for $\mathbf{\Psi}$, and also for $\mathbf{\Psi}^+_{im,x}$. In the latter case we mean a simple root for the root system $\mathbf{\Psi}_{im,x}$, as opposed to a simple root for $\mathbf{\Psi}$ which happens to be in $\mathbf{\Psi}_{im,x}$.

In Section 5 we introduce notation

$$\Psi^+_{im}[x].$$

This is a set of positive roots for $\Psi_{im,x}$; its dependence on $x$ is subtle.

# 5 Choice of Basepoints

Recall [4] $\mathcal{I} \subset W\delta$ is the set of twisted involutions, and $p : \mathcal{X} \to \mathcal{I}$. We write $u\cdot$ for the action of $u \in W$ on $\mathcal{X}$ and $\mathcal{I}$. Also recall we have chosen an element $\delta$ so that $B$ is large with respect to $\theta_\delta$.

We need a result of the following form. Recall $\mathcal{X}(w) = \{x \in \mathcal{X} \mid p(x) = w\}$. For $w \in \mathcal{I}$ let

$$\mathcal{X}(w,\delta) = \{x \in \mathcal{X}(w) \mid x \text{ is conjugate to } \delta\} \tag{5.1}$$

This set is not empty [6, Lemma 9.17]. Recall the restriction of $\theta_x$ to $H$ only depends on the image of $x$ in $\mathcal{I}$, and we write $\theta_x = \theta_w$ accordingly. As usual let $\Psi_{im,w}$ be the imaginary roots with respect to $\theta_w$ and let $W_{im,w} = W(\Psi_{im,w})$. Then $\mathcal{X}(w,\delta)$ a single $W_{im,w}$-orbit.

For $x \in \mathcal{X}$ let

$$\mathcal{X}(x,\delta) = \mathcal{X}(p(x),\delta). \tag{5.2}$$

Fix $\theta$. We say a set $\Psi^+$ of positive roots is *large* if every simple root is complex or non-compact imaginary. In particular if $\Psi^+_{im}$ is a set of positive imaginary roots, it is large if every simple root is non-compact.

Fix $w$. We want a $W_{im,w}$ equivariant map which takes $x \in \mathcal{X}(w,\delta)$ to a $W(K_x,H)$-orbit of large sets $\Psi^+_{im,w}[x]$ with respect to $\theta_x$.

**Assumption 5.3** *Given $w \in \mathcal{I}$ and $x \in \mathcal{X}(w,\delta)$ we have chosen a set $\Psi^+_{im,w}[x]$ of positive imaginary roots, large with respect to $\theta_x$, and defined up to conjugacy by $W(K_x,H)$. We assume these sets satisfy*

$$\Psi^+_{im,u\cdot w}[u\cdot x] = u\Psi^+_{im,w}[x] \quad (u \in W) \tag{5.4}$$

*(up to the action of $W(K_{u\cdot w},H)$ on the left and $W(K_x,H)$ on the right).*

In particular

$$\Psi^+_{im,u\cdot w}[u\cdot x]=u\Psi^+_{im,w}[x]\quad(u\in W_{im}). \tag{5.5}$$

Given $\Psi^+_{im,w}[x]$ for a single $x\in\mathcal{X}(w,\delta)$ this defines $\Psi^+_{im}[x]$ for all $x'\in\mathcal{X}(w,\delta)$.

A natural choice of $\Psi^+_{im,w}$ is $\mathbf{\Psi}^+_{im,w}$, where $\mathbf{\Psi}^+$ is our fixed set of positive roots (cf. Section 4). This is large with respect to some small number of choices of $x\in\mathcal{X}(w,\delta)$. For example if $w=p(\delta)$ then we can take $x=\delta$. There may be other choices.

For example if $G=SL(2)$ and $w=1$ we may take $x=\pm\delta$. Of course the natural choice is $\delta$. Note that $w\delta=-\delta$, but $w(\delta,B)=(-\delta,\overline{B})$.

Condition (5.4) now takes the following form.

**Assumption 5.6** *For each $w\in\mathcal{I}$ we are given an element $x_b[w]\in\mathcal{X}(w,\delta)$ (the "basepoint") satisfying: $\mathbf{\Psi}^+_{im,w}$ is large with respect to $x_b[w]$. These satisfy the following condition. Suppose $u\in W$, and choose $\sigma\in W$ satisfying*

$$x_b[u\cdot w]=\sigma\cdot x_b[w] \tag{5.7(a)}$$

*Then*

$$\mathbf{\Psi}^+_{im,u\cdot w}=\sigma\mathbf{\Psi}^+_{im,w}. \tag{5.7(b)}$$

**Remark 5.8** For example if $u\cdot w=w$, i.e. $u\in W^{\theta_w}$, we may take $\sigma=1$, and (5.7)(a) is immediate.

**Remark 5.9** Note that $u\cdot x_b[w]\in\mathcal{X}(u\cdot w,\delta)$, which implies $\sigma=\tau u$ with $\tau\in W_{im,u\cdot w}$.

**Definition 5.10** *Given $w$ let $x_b=x_b[w]$ as in Assumption 5.6. Let*

$$\Psi^+_{im,w}[x_b]=\mathbf{\Psi}^+_{im,w} \tag{5.11(a)}$$

*Define $\Psi^+_{im,w}[x]$ for all $x\in\mathcal{X}(w,\delta)$ by (5.5). That is, given $x\in\mathcal{X}(w,\delta)$ choose $\sigma\in W(\mathbf{\Psi}_{im,w})$ satisfying $\sigma\cdot x_b=x$ and define*

$$\Psi^+_{im,w}[x]=\sigma\mathbf{\Psi}^+_{im,w}. \tag{5.11(b)}$$

*By (5.4) this defines $\Psi^+_{im,w}[x]$ (up to conjugacy by $W_{im}(K_x,H)$) for all $x\in\mathcal{X}(w,\delta)$.*

We check that in fact Definition 5.10 implies (5.4) holds.

**Lemma 5.12** *Assume the basepoints $x_b[w]$ satisfy Assumption 5.6. Then the map $x \to \Psi^+_{im,w}[x]$ of Definition 5.10 satisfies* (5.4)*.*

**Proof.** Fix $w \in I$, $x \in \mathcal{X}(w,\delta)$, and $u \in W$. We have to show

(5.13)(a) $$\Psi^+_{im,u\cdot w}[u\cdot x] = u\Psi^+_{im,w}[x]$$

For the left hand side choose $\sigma \in W(\Psi_{im,u\cdot w})$ so that

(5.13)(b) $$u\cdot x = \sigma\cdot x_b[u\cdot w].$$

Similarly choose $\tau \in W(\Psi_{im,w})$ so that

(5.13)(c) $$x = \tau\cdot x_b[w]$$

Note that these two relations imply

(5.13)(d) $$x_b[u\cdot w] = \sigma^{-1}u\tau\cdot x_b[w]$$

We have to show

(5.13)(e) $$\sigma\mathbf{\Psi}^+_{im,u\cdot w} = u\tau\mathbf{\Psi}^+_{im,w}$$

i.e.

(5.13)(f) $$\mathbf{\Psi}^+_{im,u\cdot w} = \sigma^{-1}u\tau\mathbf{\Psi}^+_{im,w}.$$

This follows from (5.13)(d) and Assumption 5.6. □

Fix $w \in \mathcal{I}$ and $x_b[w]$. Suppose $\alpha$ is a real or imaginary root with respect to $\theta_w$. Then $s_\alpha\cdot w = w$ so

$$x_b[s_\alpha\cdot w] = x_b[w].$$

(Warning: if $\alpha$ is real then $x_b[s_\alpha\cdot w] = s_\alpha\cdot x_b[w] = x_b[w]$, but this may fail if $\alpha$ is non-compact imaginary, since such $s_\alpha$ may act non-trivially on $\mathcal{X}(w)$.)

Now suppose $\alpha$ is a simple root of $\mathbf{\Psi}^+$ which is complex. It is easy to see that

(5.14) $$s_\alpha\mathbf{\Psi}^+_{im,w} = \mathbf{\Psi}^+_{im,s_\alpha\cdot w}.$$

(The main point is that the imaginary the roots of $\mathbf{\Psi}^+$ and $s_\alpha\mathbf{\Psi}^+$ are the same, since $s_\alpha$ only changes the sign of the complex root $\alpha$). This shows that in (5.7)(a-b) if $u = s_\alpha$ we may take $\sigma = s_\alpha$, i.e.

$$x_b[s_\alpha \cdot w] = s_\alpha \cdot x_b[w]$$

for $\alpha$ a complex simple root. This gives:

**Lemma 5.15** *Suppose that for all $w \in \mathcal{I}$ we are given $x \in \mathcal{X}(w,\delta)$ such that $\mathbf{\Psi}^+_{im,w}$ is large with respect to $x_b[w]$. These choices satisfy the second condition of Assumption 5.6 if the following condition holds for all $w \in I$. Suppose $\alpha$ is a simple root for $\mathbf{\Psi}^+$ and is complex with respect to $w$. Then*

$$x_b[s_\alpha \cdot w] = s_\alpha \cdot x_b[w] \tag{5.16}$$

So far these choices do two things: ensure that our definition of standard modules is independent of the choice of $\Psi^+_R$ (Definition 6.7 and Proposition 6.9), and gives the correct cross action (Proposition 6.14).

We also need some compatibility of these choices under Cayley transforms.

**Lemma 5.17** *Suppose $\mathbf{\Psi}^+_{im,w}$ is large with respect to $\theta_w$. Let $\alpha$ be an imaginary non-compact root which is simple for $\mathbf{\Psi}^+_{im,w}$. Then $\mathbf{\Psi}^+_{im,\sigma_\alpha w}$ is large with respect to $\sigma_\alpha w$.*

**Proof.** David sketched a proof of this. Becky Herb gave a complete case by case proof. □

This tells us we could choose $x_b[\sigma_\alpha w] = \sigma_\alpha x_b[w]$ with $\alpha$ as in the Lemma. We put this all together.

**Proposition 5.18** *There exists at most one choice of $\{x_b[w] \mid w \in \mathcal{I}\}$ satisfying the following conditions for all $w \in \mathcal{I}$. Define real, imaginary, complex, compact and non-compact with respect to $\theta_w$.*

1. *$x_b[p(\delta)] = \delta$.*
2. *$\mathbf{\Psi}^+_{im,w}$ is large with respect to $x_b[w]$.*
3. *Suppose $\alpha$ is a simple root of $\mathbf{\Psi}^+$. Then*

$$x_b[s_\alpha \cdot w] = \begin{cases} x_b[w] & \text{if } \alpha \text{ is real or imaginary} \\ s_\alpha \cdot x_b[w] & \text{if } \alpha \text{ is complex} \end{cases}$$

4. *Suppose $\alpha$ is imaginary, non-compact, and simple for $\Psi^+_{im,w}$. Then*

$$x_b[\sigma_\alpha w] = \sigma_\alpha x_b[w]$$

Here $\sigma_\alpha$ is an element mapping to $s_\alpha$ as in [4]. Recall this is the Cayley transform action, i.e. left multiplication by $\sigma_\alpha$.

The proof is immediate; starting at $p(\delta)$ we may get to any $w \in \mathcal{I}$ by a series of Cayley transforms and cross actions as in the Proposition.

**Conjecture 5.19** *There exists $x_b[w]$ as in the Proposition.*

Fokko has built a check of this into the atlas software.

We will apply this on the dual side.

# 6 Standard Modules

We work in the setting of [4], so we have fixed $H, B, H^\vee, B^\vee$.

**Definition 6.1 ([6], Definition 8.18)** *Fix $\theta$, a set $\Psi^+$ of positive roots, and an element $\lambda \in X^*(H) \otimes \mathbb{C}$. We say $(\Psi^+, \lambda)$ is in good position if the following conditions hold.*

1. *If $\alpha$ is real then $\langle \lambda, \alpha^\vee \rangle < 0$,*
2. *If $\alpha$ is imaginary then $\langle \lambda, \alpha^\vee \rangle > 0$,*
3. *Suppose $\alpha, \theta\alpha$ are positive complex roots. Then $\langle \lambda, \alpha^\vee \rangle > 0$ or $\langle \lambda, \theta\alpha^\vee \rangle > 0$.*
4. *Suppose $\alpha, -\theta\alpha$ are positive complex roots. Then $\langle \lambda, \alpha^\vee \rangle < 0$ or $\langle \lambda, \theta\alpha^\vee \rangle < 0$.*

*Now suppose $\Psi^+_R$ is the set of positive real roots. We say $(\Psi^+_R, \lambda)$ is in good position if $\langle \lambda, \alpha^\vee \rangle < 0$ for all $\alpha \in \Psi^+_R$.*

*If $\Lambda$ is a character we say $(\Psi^+, \Lambda)$ is in good position if $(\Psi^+, d\Lambda)$ is in good position, and similarly $\Psi^+_R, \Lambda)$.*

**Definition 6.2** *Fix $x$. Suppose $\Psi^+_{R,x}$ is an arbitrary set of positive real roots and $\Lambda$ is a genuine $H_\rho(\mathbb{R})$-module. Let $\lambda = d\Lambda$. Then the standard module*

$$I(x, \Psi^+_{R,x}, \Lambda)$$

*is defined [6, Definition 8.27].*

*Since $x$ is already specified we will sometimes drop it from the notation and write*

$$I(x, \Psi_R^+, \Lambda).$$

*We have included $x$ to indicate the strong real form; this is a $(\mathfrak{g}, K_x)$-module.*

*We can always reduce to the case $(\Psi_R^+, \Lambda)$ in good position by [6, Lemma 8.24]:*

$$I(x, \Psi_R^+, \Lambda) \simeq I(x, w\Psi_R^+, \Lambda \otimes \tau(\Psi_R^+, w)) \tag{6.3}$$

*for $w \in W_R$.*

Recall $\tau(\Psi_R^+, w)$ is a $\mathbb{Z}/2\mathbb{Z}$-valued character of $H_x(\mathbb{R})$. See (2.8) and [6, Proposition 8.24].

**Remark 6.4** In [6, Definition 8.27] there is also a choice of positive real roots $P$. Since $\lambda$ is regular $P$ is determined by $\lambda$ and we have dropped it from the notation. While $\Psi_R^+$ is also determined by $\lambda$ if it is in good position, we do not want to assume good position, so we do not drop it from the notation.

It is worth noting that

$$I(wx, \Psi_{R,wx}^+, \Lambda) \simeq I(x, w\Psi_{R,x}^+, w\Lambda) \quad (w \in W(G, H)). \tag{6.5)(a}$$

That is there is an isomorphism from $K_x$ to $K_{wx}$ taking the $(\mathfrak{g}, K_x)$-module $I(x, \Psi_R^+, \Lambda)$ to the $(\mathfrak{g}, K_{wx})$-module $I(wx, w\Psi_R^+, w\Lambda)$. In particular

$$I(x, \Psi_R, \Lambda) \simeq I(x, w\Psi_R, w\Lambda) \quad (w \in W(K_x, H)). \tag{6.5)(b}$$

**Lemma 6.6** *Suppose $(\Psi, \Lambda)$ and $(\Psi, \Lambda')$ are in good position. Then $I(x, \Psi, \Lambda) \simeq I(x, \Psi, \Lambda')$ if and only if $\Lambda \simeq w\Lambda'$ for some $w \in W(K_x, H)$ satisfing $w\Psi_R^+ = \Psi_R^+$.*

**Note:** I think this isn't right - you might need to allow moving the positive system and a $\tau$ term - see [6].

Because of this we sometimes want to put our data in good position.

Fix an infinitesimal character (regular integral) and write it as $\lambda$ for $\lambda \in X_*(H) \otimes \mathbb{C}$ $B^\vee$-dominant (identified with a $B$-dominant element of $X^*(H) \otimes \mathbb{C}$ via our fixed $\zeta$).

Now we give the crucial definition of the representation associated to L-data.

**Definition 6.7** *Suppose* $(x,y)\in\mathcal{Z}$ *and* $e^{2\pi i\lambda}=y^2$. *Choose* $y_0\in\mathcal{X}(y,\delta^\vee)$. *The set* $\Psi^+_{im,y}[y_0]$ *for* $G^\vee$ *(Assumption 5.6) corresponds to a set* $\Psi^+_{R,x}[y_0]$ *of positive real roots for* $G$. *Let* $\Psi^-_{im,y}[y_0]=-\Psi^+_{im,y}[y_0]$ *(cf. Definition 6.1(1)).*

*Define*

$$I(x,y,\lambda)=I(x,\Psi^-_{R,x}[y_0],\Lambda[x,y,\lambda,y_0]) \tag{6.8)(a}$$

*Let* $y_b=y_b[y]$, *and recall* $\Psi^+_{im,y}[y_b]=\mathbf{\Psi}^+_{R,y}$ *(Assumption 5.6). By Proposition 6.9 we may choose* $y_0=y_b$, *and we see*

$$I(x,y,\lambda)=I(x,\mathbf{\Psi}^-_{R,x},\Lambda[x,y,\lambda,y_b]) \tag{6.8)(b}$$

This is a $(\mathfrak{g},K_x)$-module, or more casually a $G_x(\mathbb{R})$-module.

The next Lemma provides one of the justifications for Assumption 5.6.

**Proposition 6.9** *The module* $I(x,y,\lambda)$ *is independent of the choice of* $y_0\in\mathcal{X}(y,\delta^\vee)$.

Implicit in this is the assertion that the module is independent of the choice of $\Psi^-_{R,x}[y_0]$, which is only defined up to conjugacy by $K^\vee_y$.

**Proof.** Let $\theta^\vee=\theta_y$, $w^\vee_0=p(y)\in\mathcal{I}$. Suppose $y'_0$ is another choice, i.e. $y'_0=w^\vee y_0$ for $w^\vee\in W^{\theta^\vee}$ (on the dual side). The action of this group on $\mathcal{X}(w^\vee_0)$ is trivial, except for the action of $W_{im}$.

We need to show for $w^\vee\in W_{im}(G^\vee,H^\vee)$, with $w=\zeta^{-1}(w^\vee)\in W_R(G,H)$

$$I(x,\Psi^-_R(w^\vee y_0),\Lambda(x,y,\lambda,w^\vee y_0))=I(x,\Psi^-_R(y_0),\Lambda(x,y,\lambda,y_0))$$

By (5.4) and (6.3)

$$\begin{aligned}I(x,\Psi^-_R(w^\vee y_0),\Lambda(x,y,\lambda,w^\vee y_0))&=I(x,w\Psi^-_R(y_0),\Lambda(x,y,\lambda,w^\vee y_0))\\&=I(x,\Psi^-_R(y_0),\Lambda(x,y,\lambda,w^\vee y_0)\otimes\tau(w))\end{aligned}$$

So it is enough to show

**Lemma 6.10**

$$\Lambda[x,y,\lambda,w^\vee y_0]=\Lambda[x,y,\lambda,y_0]\otimes\tau(w) \tag{6.11}$$

This is basically [6, Lemma 9.28].

**Proof.** It is enough to check this with $w = s_\alpha$ for $\alpha$ a simple root of $\Psi_R$. Then $\alpha^\vee$ is an imaginary root. As in Section 3 write

$$y = e^{2\pi i\tau} y_0$$

so

$$\Lambda[x, y, \lambda, y_0] = \Lambda(x, \lambda, \kappa)$$

with

$$\kappa = \lambda - (1 + \theta^\vee)\tau$$

Recall (2.9)

$$\tau(s_\alpha) = \Lambda(0, \alpha^\vee)$$

so the right hand side of (6.11) is

$$\Lambda(x, \lambda, \kappa')$$

with

$$\begin{aligned}\kappa' &= \lambda - (1 + \theta^\vee)\tau + \alpha^\vee \\ &= \lambda - (1 + \theta^\vee)(\tau + \frac{1}{2}\alpha^\vee)\end{aligned}$$

To compute $\Lambda[x, y, \lambda, s_{\alpha^\vee} y_0]$ we write $s_{\alpha^\vee} y_0 = g\theta^\vee(g^{-1})y_0$ where $g \in Norm_{G^\vee}(H^\vee)$ gives $s_{\alpha^\vee}$. Write $g\theta^\vee(g^{-1}) = e^{2\pi i\eta}$. Then

$$y = e^{2\pi i(\tau+\eta)} y_0$$

and therefore

$$\Lambda[x, y, \lambda, s_{\alpha^\vee} y_0] = \Lambda(x, \lambda, \kappa'')$$

where

$$\kappa'' = \lambda - (1 + \theta^\vee)(\tau + \eta)$$

So it is enough to show

$$\eta = \frac{1}{2}\alpha^\vee$$

i.e.

$$g\theta^\vee(g^{-1}) = \alpha^\vee(-1) = m_\alpha.$$

Now it is crucial that $y_0$ is large, so that $\alpha^\vee$ is non-compact imaginary. This is now an $SL(2)$ calculation. See the last formula on page 126 of [6].

This proves the the Lemma. $\square$

This completes the proof of the Proposition. $\square$

**Remark 6.12** In the terminology of Section 2 $\Lambda = \Lambda(x, \lambda, \nu)$ for this given $\lambda$, and some $\nu$. The main thing to keep track of is therefore $\nu$.

With this setup $x$ hardly matters at all. Its only role is to define the real forms of $G$ and $H$.

For example suppose $G$ is connected and contains a compact Cartan subgroup, and suppose $\lambda \in \rho + X^*(H)$. The discrete series with infinitesimal character $\lambda$ are the set of

$$I(x, \emptyset, \Lambda(x, \lambda, 0))$$

as $x$ varies. Each $\Lambda(x, \lambda, 0)$ is the "same" character of $H$; the only thing which changes is the real form $x$ of $H$ and $G$.

## 6.1 Cross Action

Suppose $I(x, \Psi_R^+, \Lambda)$ is a standard module and $w \in W = W(G, H)$. See Remark 1.1.

**Lemma 6.13 ([5])**

$$w \times I(x, \Psi_R^+, \Lambda) \simeq I(x, \Psi_R^+, w^{-1} \times \Lambda)$$

We need to compute this in $I(x, y, \lambda)$ coordinates.

**Proposition 6.14**

$$w \times I(x, y, \lambda) = I(wx, wy, \lambda)$$

**Proof.** Choose $y_0 \in \mathcal{X}(\delta, y)$. The left hand side is

$$\begin{aligned} w \times I(x, \Psi_R^+[y_0], \Lambda[x, y, \lambda, y_0]) &= I(x, \Psi_R^+[y_0], w^{-1} \times \Lambda[x, y, \lambda, y_0]) \\ &= I(x, \Psi_R^+[y_0], \Lambda[x, y, w^{-1}\lambda, y_0]) \end{aligned} \tag{6.15}$$

On the right hand side we may take $wy_0$ as our base point. Then we get

$$\begin{aligned} I(wx, wy, \lambda) &= I(wx, \Psi_R^-[wy_0], \Lambda[wx, wy, \lambda, wy_0]) \\ &= I(x, w^{-1}\Psi_R^-[wy_0], w^{-1}\Lambda[wx, wy, \lambda, wy_0]) \\ &= I(x, w^{-1}\Psi_R^-[wy_0], \Lambda[x, y, w^{-1}\lambda, y_0]) \end{aligned} \tag{6.16}$$

The result follows from (5.4), which says that $w^{-1}\Psi_R^-[wy_0] = \Psi_R^-[y_0]$. $\square$

## 6.2 Parity Condition and the Cross Stabilizer

Suppose $\alpha$ is a real root. Then $s_\alpha x = x$. We see that a real root $\alpha$ is in the cross stabilizer of $\pi = I(x, \Psi_R^-, \Lambda)$, i.e. $s_\alpha \times \pi \simeq \pi$, if and only if

$$s_\alpha y = y.$$

This is on the dual side, and says that $s_\alpha \in W(K_y^\vee, H^\vee)$, $s_\alpha$ is an imaginary reflection, in the cross stabilizer of the dual representation.

It is useful to compute this directly in terms of $I(x, \Psi_R^-, \Lambda)$, if only as a sanity check. Recall the parity condition [8]. The roots which do not satisfy the parity condition are the + part of the cograding of real roots; I like to think of these as the "non-parity" roots, also known as the irreducibility roots. Then $\alpha$ fails the parity condition implies $s_\alpha \times \pi = \pi$, but not conversely.

**Lemma 6.17** *Fix $\pi = I(x, \Psi_R^-, \Lambda)$ and a real root $\alpha$. Write $\Lambda = \Lambda(x, \lambda, \nu)$ and let $k = \langle \lambda - \nu - \rho_R, \alpha^\vee \rangle$.*

1. *$s_\alpha \times \pi \simeq \pi$ if and only if $sgn(\alpha)^k = 1$*
2. *$\alpha$ is a non-parity root if and only if $k \equiv 0 \pmod 2$.*
3. *$\alpha$ is a non-parity root if and only if $(\Lambda \otimes \rho)(m_\alpha) = (-1)^{\langle \lambda + \rho_{cx}, \alpha^\vee \rangle}$.*

**Proof.** We have

$$s_\alpha(s_\alpha \times I(x, \Psi_R^-, \Lambda)) = I(x, s_\alpha \Psi_R^-, s_\alpha(s_\alpha \times \Lambda))$$

which by (6.3) equals

$$I(x, \Psi_R^-, s_\alpha(s_\alpha \times \Lambda)) \otimes \tau(\Psi_R^-, \alpha)$$

By (2.18) we have

$$\begin{aligned} s_\alpha(s_\alpha \times \Lambda(x, \lambda, \nu)) \otimes \tau(s_\alpha) &= \Lambda(x, \lambda, \nu) \otimes \Lambda(0, \langle \lambda - \nu + \rho_R, \alpha^\vee \rangle \alpha) \\ &= \Lambda(x, \lambda, \nu) \otimes \mathrm{sgn}(a)^k. \end{aligned} \tag{6.18}$$

This proves the first claim.

If $\alpha$ is type $II$, i.e. $\mathrm{sgn}(\alpha) = 1$, then obviously $s_\alpha \times \pi = \pi$. If $\alpha$ is type $I$ then this holds if and only if $k \equiv 0 \pmod 2$. This proves (2). [Check: this is really correct.]

Now
$$\langle \lambda - \nu - \rho_R, \alpha^\vee \rangle \in 2\mathbb{Z}$$
if and only if
$$\langle \lambda + \rho_{im} + \rho_{cx} - (\nu + \rho), \alpha^\vee \rangle \in 2\mathbb{Z}$$
Here $\rho = \rho_R + \rho_{im} + \rho_{cx}$ as usual. Now $\Lambda \otimes \rho$ factors to $H$ and it is easy to see that
$$(\Lambda \otimes \rho)(m_\alpha) = (-1)^{\langle \nu + \rho, \alpha^\vee \rangle}$$
On the other hand $\langle \rho_{im}, \alpha^\vee \rangle = 0$ and therefore
$$(-1)^{\langle \lambda + \rho_{im} + \rho_{cx} - (\nu + \rho), \alpha^\vee \rangle} = (-1)^{\langle \lambda + \rho_{im}, \alpha^\vee \rangle} \Lambda(m_\alpha)$$

□

The parity condition in this result is simpler than that of [8], it doesn't have the mysterious $\epsilon_\alpha$ term. The relation involves the next result, cf. [2, Lemma 14.6]. Also see Section 7.

**Lemma 6.19** *Suppose $\Psi^+$ satisfies: $\alpha > 0$ complex implies $\theta\alpha < 0$. Let $\rho_{cx} = \frac{1}{2}\sum_{\alpha \in \Psi^+_{cx}} \alpha$ where $\Psi^+_{cx}$ is the positive complex roots. Then $\epsilon_\alpha = (-1)^{\langle \rho_{cx}, \alpha^\vee \rangle}$.*

**Remark 6.20** If $\Psi^+$ satisfies the conditions of the Lemma then $\Lambda \otimes \rho$ is what is denoted $\Gamma$ elsewhere (cf. Section 7), but not otherwise.

## 6.3 Cayley Transforms

Fix $x$ let $\Psi_{R,x}$ be the real roots with respect to $\theta_x$, and choose a set of positive real roots $\Psi^+_{R,x}$. Suppose $I(x, \Psi^-_{R,x}, \Lambda)$ is a standard module.

Now suppose $\alpha$ is a non-compact imaginary root and $\Psi^+_{R,c^\alpha x}$ is a set of positive real roots for $\theta_{c^\alpha x}$. Then $I(\sigma_\alpha x, \Psi^-_{R,\sigma_\alpha x}, c^\alpha \Lambda)$ is defined.

Assume $(\Psi^+_{R,x}, \Lambda)$ and $(\Psi^+_{R,\sigma_\alpha x}, c^\alpha \Lambda)$ are in good position. For example take $\mathbf{\Psi}^+_{R,x}$ and $\mathbf{\Psi}^+_{R,\sigma_\alpha x}$. [Actually, you need to take the opposite root system... there is something to straighten out here.]

**Proposition 6.21**

$$c^\alpha I(x, \Psi^-_{R,x}, \Lambda) = I(\sigma_\alpha x, \Psi^-_{R,\sigma_\alpha x}, c^\alpha \Lambda)$$

**Note:** To be honest I'm guessing here - this could be checked from the usual definition. I've partly gotten it by working backwards. I'm not sure about the "good position" aspect.

**Proposition 6.22** *Suppose $\alpha$ is a noncompact imaginary root of $\Psi$ with respect to $x$.*

$$c^\alpha I(x,y,\lambda) = I(\sigma_\alpha x, \sigma^\alpha y, \lambda) \tag{6.23}$$

**Proof.**

We first reduce to the case $\alpha$ is simple for $\mathbf{\Psi}^+_{im,x}$. If $\alpha$ is not simple choose $w \in W_{im}$ so that $\beta = w\alpha$ is simple.

$$\begin{aligned} c^\alpha I(x,y,\lambda) &= w \times c^\beta I(x,y,\lambda) \\ &= w \times I(\sigma_\beta x, \sigma^\beta y, \lambda) \text{ (assuming the result for simple roots)} \\ &= I(w\cdot\sigma_\beta x, w\cdot\sigma^\beta y, \lambda) \quad \text{(Proposition 6.14)} \\ &= I(\sigma_\alpha x, \sigma^\alpha y, \lambda) \end{aligned}$$

So assume $\alpha$ is simple for $\mathbf{\Psi}^+_{im,x}$. Choose $y_0 = y_0[y]$, so

$$\Psi^+_R[y_0] = \mathbf{\Psi}^+_{R,y}.$$

The left hand side is then

$$\begin{aligned} c^\alpha I(x,y,\lambda) &= c^\alpha I(x, \mathbf{\Psi}^+_{R,x}, \Lambda[x,y,\lambda,y_0] \\ &= I(\sigma_\alpha x, \mathbf{\Psi}^+_{R,\sigma_\alpha x}, \Lambda[\sigma_\alpha x, \sigma^\alpha y, \lambda, \sigma^\alpha y_0]) \end{aligned}$$

The last equality uses the fact that $(\mathbf{\Psi}^+_{R,\sigma_\alpha x}, \Lambda)$ is in good position (this only depends on $\lambda = d\Lambda$).

On the other hand choose $y_0' = y_0[\sigma^\alpha y]$, so the right hand side is

$$I(\sigma_\alpha x, \sigma^\alpha y, \lambda) = I(\sigma_\alpha x, \mathbf{\Psi}^+_{R,\sigma_\alpha x}, \Lambda[\sigma_\alpha x, \sigma^\alpha y, \lambda, y_0'])$$

So we are done provided $y_0' = \sigma^\alpha y_0$, i.e.

$$y_0[\sigma^\alpha y] = \sigma^\alpha y_0[y]$$

Writing this on the $G$ side, and with $w = p(y)$ in place of $y$ we need

$$x_b[\sigma_\alpha w] = \sigma_\alpha x_b[w] \tag{6.24}$$

for $\alpha$ an imaginary noncompact root, simple for $\mathbf{\Psi}^+_{im,w}$. This is precisely Proposition 5.18, cf. Conjecture 5.19. $\square$

# 7 Relating $\Gamma$ and $\Lambda$

This is taken from [7]. We relate $I(\Psi_R^+,\Lambda)$ to $I(\Gamma,\lambda)$. Here $(\Gamma,\lambda)$ is a regular character as in [8]. We don't need final limit parameters here.

Given $I(\Psi_R^+,\Lambda)$, choose any set $\Psi^+$ of positive roots satisfying $\Psi_R^+\subset\Psi^+$ and

$$\alpha>0 \quad \text{complex} \Rightarrow \theta\alpha<0. \tag{7.1}$$

Let $\rho=\rho(\Psi)$, and let $\rho_{im,c}$ be one half the sum of the positive imaginary compact roots. Write $\Lambda=\Lambda(\lambda,\nu)$. Let

$$\Gamma=\Lambda(\lambda+\rho_i-2\rho_{im,c},\nu+\rho-2\rho_{im,c})$$

According to [7] then have $I(\Lambda,\Psi_R^-,\lambda)=I(\Gamma,\lambda)$.

**Remark 7.2** Check this. Is the $\Psi_R^+$ correct?

# References


[1] J. Adams. *Lifting of characters*, volume 101 of *Progress in mathematics*. Birkhäuser, Boston,Basel,Berlin, 1991.

[2] J. Adams and P. Trapa. Duality for nonlinear simply laced groups.

[3] Jeffrey Adams. Parameters for representations of real groups. preprint, atlas web site.

[4] Fokko du Cloux. Combinataorics for the representation theory of reductive groups. preprint.

[5] D. Vogan J. Adams. Lifting of characters and Harish-Chandra's method of descent. preprint.

[6] D. Vogan J. Adams. L-groups, projective representations, and the Langlands classification. 113:45–138, 1992.

[7] Annegret Paul. The dictionary of representations. preprint.

[8] D. Vogan. Irreducible characters of semisimple Lie groups IV. character-multiplicity duality. 49, No. 4:943–1073, 1982.